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Bifurcations of Certain Family of Planar Vector Fields Tangent to Axes

HENRYK ŻOŁĄDEK

Institute of Mathematics, Warsaw University, 00-901 Warsaw, PKiN, IX p., Poland

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The topological versality of the family V_μ,

$$\begin{aligned} \dot{x} &= x(\mu_1 + x + y), \\ \dot{y} &= y(\mu_2 + ax + by + cy^2), \end{aligned} \tag{1}$$

of vector fields on the plane is proven. The asymptotic formulas of the bifurcational curves are given. The proofs rely on an analysis of the limit cycles and the non-abelian integrals along irrational curves.

1. INTRODUCTION

The family (1) is the last case of a two parameter family of vector fields on the plane, which arises from natural problems in bifurcation theory of dynamical systems and versality of which has not yet been established.

The families of vector fields on the plane which are invariant with respect to the cyclic group $\mathfrak{C}_q$ generated by the rotation of the plane throughout an angle $2\pi/q$, $q = 1, 2, \ldots$, have been considered by Bogdanov [7, 8] ($q = 1$), Arnold [2, 3], Iliashenko [27] ($q = 1, 2, 3$), Khorozov [30] ($q = 2, 3$), Takens [40], Neishtat [35] ($q = 4$), Berezovskaya and Khibnik [6] ($q = 4$) and Wan [42] ($q = 4$). The case $q = 4$ has not been solved completely, the difficult cases have been treated only numerically (see [6]).

We note also the recent result of Dumotier, Roussarie, and Sotomayor [18], where the versality of a certain 3-parameter family of $\mathfrak{C}_1$-symmetric vector fields is proven.

The families of vector fields which are invariant with respect to the dihedral group $\mathfrak{D}_1$ generated by reflection with respect to a line has been investigated by Gavrilov [20], by the author [45], by Guckenheimer [24], and by Carr, Chow, and Hale [11].

In the present paper we consider the families of vector fields invariant with respect to the dihedral group $\mathfrak{D}_2$ (generated by two reflections with respect to two orthogonal axes). (Change $(x, y) \rightarrow (x^2, y^2)$ shows that (1) describes such families). Unexpectedly, in the considered problem the

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dihedral group $\mathfrak{D}_3$ or the group of permutations of three elements appears. This fact is essential in the proof of the main result of this work.

The author does not know any natural model in dynamical systems which leads to the vector fields invariant with respect to the finite groups $\mathfrak{D}_q$, $q \geqslant 3$ [14].

Takens [40] and Arnold [2, 3] have shown how to obtain $\mathfrak{C}_q$-invariant vector field from the periodic orbit in $\mathbb{R}^3$ with $\exp(\pm 2\pi i p/q)$ as characteristic multipliers. The $\mathfrak{D}_1$ case is connected with the deformations of vector fields in $\mathbb{R}^3$ with 0 and $\pm i\omega$ as eigenvalues of linear part at singular point (see [20, 13, 24, 25]). We sketch shortly how to obtain the case $\mathfrak{D}_2$ (see [13, 21, 25]).

Let

$$\begin{aligned} \dot{x}_1 &= \mu_1 x_1 + \omega_1 x_2 + O(|x|^2), \\ \dot{x}_2 &= -\omega_1 x_1 + \mu_1 x_2 + O(|x|^2), \\ \dot{x}_3 &= \mu_2 x_3 + \omega_2 x_4 + O(|x|^2), \\ \dot{x}_4 &= -\omega_2 x_3 + \mu_2 x_4 + O(|x|^2), \end{aligned} \tag{2}$$

be a two paramater family of vector fields in $\mathbb{R}^4$ with small (μ_1, μ_2), $\omega_1 \neq 0$, and $\omega_2 \neq 0$. If we choose the variables $\varphi_1 = \operatorname{arc}\ \tan(x_2/x_1)$, $\varphi_2 = \operatorname{arc}\ \tan(x_4/x_3)$, $x = x_1^2 + x_2^2$, $y = x_3^2 + x_4^2$ then the system (2) becomes the small perturbation of the system

$$\dot{x} = \dot{y} = 0, \qquad \dot{\varphi}_1 = -\omega_1, \qquad \dot{\varphi}_2 = -\omega_2.$$

Applying to the perturbed system the averaging procedure [3] we get the vector field

$$\begin{aligned} \dot{x} &= x(\mu_1 + p_1 x + p_2 y + \cdots), \\ \dot{y} &= y(\mu_2 + p_3 + p_4 y + \cdots), \end{aligned} \tag{3}$$

considered in the domain $x \geqslant 0$, $y \geqslant 0$.

The singularities of the system (3) give some approximate picture of bifurcations of the system (2). However, there are many phenomena in the system (3) which are not reflected on the system (3). Among them we distinguish the closed orbits on two- and three-dimensional tori and the intersections of stable and unstable submanifolds of singular sets. Thus the correspondence between systems (2) and (3) is not exact.

The system (3) restricted to the domain $x \geqslant 0$, $y \geqslant 0$ has been investigated by Holmes [26], Iooss and Langford [28], Gavrilov [21], and by a student of Arnold, Shvetzov [39] (see also [13] and [25]). In these works, there has been obtained the wrong partition of the space

$\{(p_1, p_2, p_3, p_4)\}$ of parameters into components of different behaviour of the system (3): 11 components in [21] and 12 components in [25] and [39]. In fact, its number is 13. Moreover, these works give no answer to a question of the number of limit cycles of the system (3).

In the present paper, we consider the "amplitude system," i.e., the system (3) considered ina whole neighbourhood of the origin in $\mathbb{R}^2$. The amplitude system

$$\begin{aligned} \dot{x} &= x(\mu_1 + p_1 x + p_2 y), \\ \dot{y} &= y(\mu_2 + p_3 x + p_4 y), \end{aligned} \tag{4}$$

has been investigated by Serebriakova [38] (in a whole plane) and by Bautin [4]. Serebriakova has considered the noninteresting case from the bifurcation theory point of view: $\mu_1 p_1 p_2 \neq 0$, $\mu_2 = p_2 = 0$, $p_2 = p_3 = 0$, $p_2 = p_4 = 0$. Bautin has showed that the system (4) has no limit cycles.

The system (3) has many applications: in the theory of electric generators [34], in the theory of gyroscope [9], in mechanics [10, 29], in astrophysics [12], in chemical kinetics [19], in biology [33, 44] and in others. Especially known is the application of the system (3) in the animal population dynamics. For this reason, the system (3) is called sometimes a "Generalized Lotka–Volterra System."

The main problem in the proof of versality of families such as (1) is the problem of uniqueness of the limit cycle appearing in the family. (Such a cycle corresponds to an invariant three-dimensional torus of the system (2),) This problem can be reduced to the problem of estimating the number of zeroes of the function of the following type

$$h \to \iint_{H \leqslant h} P \, dx \, dy, \tag{5}$$

where $P(x, y)$ and $H(x, y)$ are some functions (polynomial or not). When P and H are polynomials the problem is known as a "weakened 16th Hilbert problem" [3] and only recently Varchenko [43], Khovansky [31], and Petrov [36] have obtained certain general results on this field. In the $\mathfrak{C}_q$-symmetric cases, P and H turn out to be low order polynomials, and investigation of the function (5) has been done by Arnold [2], Bogdanov [7], Iliashenko [27], Neishtat [35], Howard and Kopell [32] (an alternate treatement of the case $q = 1$), and Roussarie [18]. In the $\mathfrak{D}_1$ and $\mathfrak{D}_2$ cases P and H are not polynomials and investigation of the corresponding function (5) was done by author in [45] and in the present paper. In the proof presented here, we use some ideas from [45]. Another proof in the case $\mathfrak{D}_1$ with some restrictions was given by Carr, Chow, and Hale in [11] (see also [22] and [37]). After completion of the present work, the author has learned that van Gils, Carr, and Sanders [23] have solved the problem of the uniqueness of the limit cycle for $\mathfrak{D}_2$ case with some symmetry restrictions ($\alpha = \beta$, see (8) below) using a completely different method.

It is remarkable that some bifurcational diagrams of the family (1) depend on the diameter of the domain in which the vector field is considered. The asymptotic behaviours of respective bifurcational curves depend on the diameter of this domain in a way that changes as a change of parameters a and b. This phenomenon is very similar to that one described in [45] (see Theorem 2a).

The outline of the paper is a follows: Section 2 contains definitions and a formulation of the results. In Section 3, we investigate the algebraic properties of germs of vector fields tangent to axes. In Section 4, we analyze the singular points and their bifurcations. In this section we give also a correct partition of the space of systems (3) considered in the domain $x \geqslant 0$, $y \geqslant 0$. Section 5 is devoted to the study of periodic orbits. The analysis of the integral (5) we put off to Section 6. The Appendix includes the proofs of three technical lemmas.

The present work had risen independently from the works of other authors working on this problem, and, after completing the results, the author learned about other papers. The author thanks Yu. S. Iliashenko, F. Takens, M. Medved. R. Roussarie, S. A. van Gils, and the review of *J. Differential Equations* for their interest in this work and for their help in the completion of the reference list.

2. Statement of the Results

In the whole paper, we consider the germs at $0 \in \mathbb{R}^2 \times \mathbb{R}^2$ of the families of vector fields of the following form (tangent to axes)

$$V_\mu(x, y) = xW_1(x, y; \mu)\, \partial_x + yW_2(x, y; \mu)\, \partial_y, \tag{6}$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is a parameter.

Definition 1. Two families V_μ and $\tilde{V}_\mu$ of the form (6) are called C^r-orbitally equivalent, $r \geqslant 0$, iff there exists a germ of C^r-diffeomorphisms

$$h: \quad (\mathbb{R}^2 \times \mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R}^2, 0),\ h = (\tilde{h}_\mu(x, y); \tilde{\tilde{h}}(\mu)), \tag{7}$$

where $\tilde{h}_\mu = (xh_1(x, y; \mu), yh_2(x, y; \mu))$ or $\tilde{h}_\mu = (yh_1(x, y; \mu), xh_2(x, y; \mu))$ are such that $\tilde{h}$ transforms the oriented integral curves of V_μ onto the oriented integral curves of $\tilde{V}_{\tilde{\tilde{h}}(\mu)}$ (i.e., V_μ and $\tilde{V}_{\tilde{\tilde{h}}(\mu)}$ are C^r-orbitally conjugated by means of $\tilde{h}_\mu$).

DEFINITION 2. The vector field V_0 $(\mu=0)$ is called *singular* iff $dV_0(0)=0$.

DEFINITION 3. The family V_μ

$$\begin{aligned} \dot{x} &= x(\mu_1+x+y), \\ \dot{y} &= y\left(\mu_2-\frac{\alpha+1}{\beta}x-\frac{\alpha}{\beta+1}y+\nu R(x+\mu_1, y)\right), \end{aligned} \tag{8}$$

where

$$R(x, y)=\frac{x^2}{\beta}+\frac{2xy}{\beta+1}+\frac{y^2}{\beta+2} \tag{9}$$

and $\alpha\leqslant\beta$, $\alpha\beta(\alpha+1)(\beta+1)(\alpha+\beta+1)\neq 0$ and $\nu=\pm 1$ if $(\alpha, \beta)\in\{\alpha>0\}\cup\{\beta<0, \alpha+\beta>-1\}\cup\{\alpha+\beta<-1, \beta>0\}$ or $\nu=0$ in other cases, is called *main family*. (This special choice of perturbation R we need for the purposes of study of limit cycles in Section 5.)

The main result of the present work is

THEOREM 1. (a) *The space of germs of deformations of singular vector fields is divided into degenerate germs and nondegenerate germs such that the degenerate germs form a union of a finite number of submanifolds of codimension one.*

(b) *Every nondegenerate deformation is C^0-orbitally equivalent to the main family.*

(c) *The partition of the half-plane $\alpha\leqslant\beta$ defined by lines $\alpha=0,-1$, $\beta=0,-1$, and $\alpha+\beta=-1$ and the partition of ν's into three values* 0, ± 1 *give the partition of the set of main families into* 11 *C^0-orbital equivalence classes* (*see Fig.* 1).

The next theorem describes the bifurcational diagrams and the phase portraits of the main family (8). Since, in the case $\nu=\pm 1$ and $\alpha+\beta<0$ (the domains VII and VIII in Fig. 1), the system (8) has a limit cycle which may become very large, we localize it by restricting considerations to a chosen a priori domain.

Denote

$$\mathscr{U}_\varepsilon=\left\{(x, y)\colon |x|<\varepsilon,\ |y|<2\varepsilon\max\left(\left|\frac{\beta+1}{\beta}\right|, \left|\frac{\beta+1}{\alpha+\beta+1}\right|\right)\right\}$$

and $\hfill (10)$

$$\mathscr{O}_\varepsilon=\left\{\mu\colon |\mu|=\sqrt{\mu_1^2+\mu_2^2}<\tfrac{1}{2}\varepsilon\min\left(1, \left|\frac{\alpha+\beta+1}{\alpha}\right|\right)\right\}, \ \varepsilon>0.$$

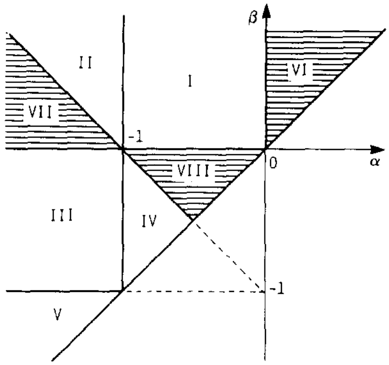

FIGURE 1

($\mathcal{U}_\varepsilon$ and $\mathcal{O}_\varepsilon$ are chosen in such a way that singular points of V_μ belong to $\mathcal{U}_\varepsilon$, $\mu \in \mathcal{O}_\varepsilon$.)

THEOREM 2. (a) *The bifurcational diagram of the family* (8) *consists of:*

(i) 4 *curves* K: $\mu_1 = 0$, L: $\mu_2 = 0$,

$$M: \quad (\beta+1)\,\mu_2 = -\alpha\mu_1 + O(\mu_1^2)$$

and

$$N: \quad \beta\mu_2 = -(\alpha+1)\,\mu_1 + O(\mu_1^2)$$

as $\mu_1 \to 0$ *in the case when* (α, β) *belongs to the domains* I–V (*Fig.* 1);

(ii) 6 *curves* K, L, M, N, Q: $\beta\mu_2 = -\alpha\mu_1 + O(\mu_1^3)$ *and* S: $\mu_2 = \kappa(\mu_1, \varepsilon)$ *in other cases* (VI–VIII). *Here, if* $\alpha+\beta > 0$ *then*

$$\kappa(\mu_1, \varepsilon) = -\frac{\alpha}{\beta}\,\mu_1 - \frac{(\beta+1)\,\mu_1^2}{\beta^2(\alpha+\beta+1)(\alpha+\beta+2)} + O(\mu_1^3) \tag{11}$$

as $\mu_1 \to 0$ *and* κ *does not depend on* ε *and if* $\alpha+\beta<0$ *then there exists the representation* $\kappa(\mu_1, \varepsilon) = \tilde{\kappa}(\mu_1/\varepsilon, \varepsilon)$, *where* $\tilde{\kappa}$ *is a continuous function satisfying*

$$\tilde{\kappa}\left(\frac{\mu_1}{\varepsilon}, \varepsilon\right) = -\frac{\alpha}{\beta}\,\mu_1 - vG\varepsilon^2 s\left(\frac{|\mu_1|}{\varepsilon}\right)(1+o(1))$$

$$\text{as} \quad \frac{|\mu_1|}{\varepsilon} + \varepsilon \to 0. \tag{12}$$

The functions G *and* s *are given in Table* I.

TABLE I

Values of $\alpha+\beta$	$s(\lambda)$	G
$(-1, 0)$	$\lambda \ln^{-1}\left(\frac{1}{\lambda}\right)$	$\frac{-\alpha(\beta+1)}{\alpha+\beta+1}\iint_{\Delta} x^{\alpha-1}y^{\beta-1}(x-y)^2\,dx\,dy$
$(-2, -1)$	$\lambda^{-\alpha-\beta}$	$\left(\frac{\beta}{\beta+1}\right)^{\beta}\frac{\Gamma(1-\alpha)}{\Gamma(\beta+1)\,\Gamma(-\alpha-\beta)}\iint_{\Delta} x^{\alpha-1}y^{\beta-1}(x-y)^2$
-2	$\lambda^2 \ln\left(\frac{1}{\lambda}\right)$	$(\beta+1)\,\beta^{-2}$
$(-\infty, -2)$	λ^2	$\frac{\beta+1}{\beta^2(\alpha+\beta+1)(\alpha+\beta+2)}$

(b) *The bifurcations of the vector field V_μ* (8) *at the curves K, L, M, N are the bifurcations of topological types of* 4 *singular points of V_μ (saddle-node bifurcations).*

(c) *If $v=1$ and the parameter μ changes from the curve Q to the curve S then in the vector field V_μ creates a unique repelling cycle Γ from the focus losing stability at Q. If $\mu \in S$ and $\alpha+\beta>0$ then Γ forms the limit contour consisting of "separatrices" of three saddles. If $\mu \in S$ and $\alpha+\beta<0$ then Γ touches the boundary of $\mathscr{U}_\varepsilon$.*

(d) *The transformation $(t, x, y; \mu) \to -(t, x, y; \mu)$ carries the family* (8) *with $v=-1$ into the family* (8) *with $v=1$. Γ is the Euler gamma funtion,*

$$\Delta = \left\{(x, y): x \geqslant 0,\ y \geqslant 0,\ x^\alpha y^\beta\left(\frac{y}{\beta+1}-\frac{x}{\beta}\right) \leqslant -(\beta(\beta+1))^{-1}\right\}.$$

From Theorems 1 and 2 we obtain the following information about the system (2) in $\mathbb{R}^4$.

COROLLARY 1. *In the generic position, the system* (2) *has at most one smooth invariant three dimensional torus.*

We shall not draw the bifurcational diagrams and the phase portraits of the family (8) because of their great number. We have 8 bifurcational diagrams divided into 8 or 12 domains and with each domain is connected a corresponding phase portrait. Thus, the whole number of the pictures is $5\cdot(1+8)+3\cdot(1+12)=84$. They are simple in fact and the reader can complete this part of the present work himself (see Theorem 2 and formula (32)).

The detailed analysis of the system (4) restricted to the domain $x \geqslant 0$, $y \geqslant 0$ is given in Section 4D.

3. Invariants, Genericity Conditions and Normal Form of the Families of Vector Fields Tangent to Axes

A. *Action of Orbital Equivalence Group on 3-Jets and Its Invariants*

Let us denote by $\mathscr{F}$ the space of germs at the point $0 \in \mathbb{R}^2$ of vector fields of the form (6) (tangent to axes) and by $J^r\mathscr{F}$, $r \geqslant 0$ denote the space of their r-jets at the point $0 \in \mathbb{R}^2$. j^rV denotes the r-jet of the vector field V at 0.

On the space $\mathscr{F}$ acts on the group of C^∞-orbital equivalences

$$\mathscr{G} = \{(h, f): h \text{ is a germ of } C^\infty\text{-diffeomorphism preserving the set } xy=0, f \text{ is a germ of } C^\infty\text{-function}, f(0)>0\}$$

by the formula

$$V \to f((h_* V)\, h^{-1}). \tag{13}$$

On the space $J^3\mathscr{F}$, this action reduces to the action of

$$J^3\mathscr{G} = \{(h, f): h \text{ is of order } 3, f \text{ is of order } 2\}$$

by the formula

$$V \to j^3(fh_* Vh^{-1}).$$

The singular vector fields form a codimension 2 subspace in $\mathscr{F}$,

$$Q^2 = \{V = xW_1\partial_x + yW_2\partial_y : W_1(0) = W_2(0) = 0\} \tag{14}$$

invariant with respect to the action of $\mathscr{G}$. Now, we restrict our attention to the vector fields (6) belonging to j^3Q^2 (singular 3-jets). They are of the form

$$\begin{aligned} V = {}& x(p_1x + p_2y + q_1x^2 + q_2xy + q_3y^2)\,\partial_x \\ & + y(p_3x + p_4y + q_4x^2 + q_5xy + q_6y^2)\,\partial_y. \end{aligned} \tag{15}$$

The action of $\mathscr{G}$ on this space reduces to the action of the follosubgroup $\mathscr{G}_0$ of $\mathscr{G}$,

$$\mathscr{G}_0 = \mathbb{Z}_2 \times \tilde{\mathscr{G}}_0, \tag{16}$$

where the generator of the group $\mathbb{Z}_2$ is $(h(x, y), f) = ((y, x), 1)$ and

$$\begin{aligned} \tilde{\mathscr{G}}_0 = \{ & ((\lambda_1^{-1}x, \lambda_2^{-1}y), \lambda_3) \circ ((x(1+\omega_1x+\omega_2y), y \\ & \times (1+\omega_3x+\omega_4y)), 1+\omega_5x+\omega_6y) : \lambda_1\lambda_2 \neq 0, \lambda_3 > 0\}. \end{aligned} \tag{17}$$

The elements of j^3Q^2 we denote by (p, q) or by V and the elements of $\tilde{\mathscr{G}}_0$ we denote by (λ, ω).

It is easy to check that $\tilde{\mathscr{G}}_0$ acts on j^3Q^2 linearly and that

$$(\lambda, \omega)(p, q) = \begin{bmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ C(\omega) & 1 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix}, \tag{18}$$

where

$$A(\lambda) = \operatorname{diag}(\lambda_1\lambda_3, \lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_2\lambda_3),$$
$$B(\lambda) = \operatorname{diag}(\lambda_1^2\lambda_3, \lambda_1\lambda_2\lambda_3, \lambda_2^2\lambda_3, \lambda_1^2\lambda_3, \lambda_1\lambda_2\lambda_3\ \lambda_2^2\lambda_3),$$

and (19)

$$C(\omega)\, p = \tilde{C}(p)\, \omega = \begin{bmatrix} 0 & 0 & 0 & 0 & p_1 & 0 \\ -p_2 & p_1 - p_3 & p_2 & 0 & p_2 & p_1 \\ 0 & -p_4 & 0 & p_2 & 0 & p_2 \\ p_3 & 0 & -p_1 & 0 & p_3 & 0 \\ 0 & p_3 & p_4 - p_2 & -p_3 & p_4 & p_3 \\ 0 & 0 & 0 & 0 & 0 & p_4 \end{bmatrix} \cdot \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix}.$$

The first result concerning this action is the computation of its invariants. (We call an invariant such a function I on j^3Q^2 that $I \circ (\lambda, \omega) = \chi(\lambda, \omega) \cdot I$, where χ is a character of the group $\tilde{\mathscr{G}}_0$ [15].)

PROPOSITION 3.1 (a) *The ring of relative polynomial invariants of the action of $\tilde{\mathscr{G}}_0$ on j^3Q^2 is generated by*

$$\begin{aligned} &p_1,\ p_3 && \textit{with weight } \lambda_1\lambda_3, \\ &p_2,\ p_4 && \textit{with weight } \lambda_2\lambda_3, \end{aligned} \tag{20}$$

and

$$\begin{aligned} I_1(p, q) = {} & (2p_2p_3 - p_1p_4 - p_3p_4)\, p_2p_4q_1 + p_1p_3p_4(p_4 - p_2)\, q_2 \\ & + p_1p_3p_4(p_1 - p_3)\, q_3 + p_1p_2p_4(p_4 - p_2)\, q_4 \\ & + p_1p_2p_4(p_1 - p_3)\, q_5 + p_1p_3(2p_2p_3 - p_1p_2 - p_1p_4)\, q_6 \\ & \textit{with weight } \lambda_1^3\lambda_2^3\lambda_3^5. \end{aligned}$$

(b) *Every absolute rational invariant (with weight $\chi \equiv 1$) is a rational function of $a = p_3/p_1$ and $b = p_4/p_2$.*

Proof. (a) Obviously, by (18) and (19) the functions p_i, $i = 1, 2, 3, 4$, are invariants. Therefore, it is enough to show that every invariant I is of the form $I=\sum a_k(p)\, I_1^k$, where a_k are some polynomials of p, $a_k \in \mathbb{R}[p]$. We concentrate our attention at the action of the subgroup $\tilde{\tilde{\mathscr{G}}}_0 = \{(\lambda, \omega) \in \tilde{\mathscr{G}}_0 \colon \lambda = \lambda_0 = (1, 1, 1)\}$, which is isomorphic to the additive group $\mathbb{R}^6$. Because any nontrivial character of $\tilde{\tilde{\mathscr{G}}}_0$ is a transcendental function of ω the invariant I is an absolute invariant of $\tilde{\tilde{\mathscr{G}}}_0$, $I \circ (\lambda_0, \omega) \equiv I$. To study the invariants we investigate the orbits of the elements of $\mathbb{R}^6 = \{(0, q) \in j^3 Q^2\}$ under the action of $\tilde{\tilde{\mathscr{G}}}_0$. More precisely, we consider the orbits of the elements of the module $(\mathbb{R}[p])^6 = \{\sum b_i(p)\, q_i\}$ under the action of $\tilde{\tilde{\mathscr{G}}}_0$ (see (17)). The orbit of $r \in \mathbb{R}[p]^6$ is equal to $\{r + \tilde{C}(p)\, \omega \colon \omega \in \mathbb{R}^6\}$ and can be parametrized by elements of the module $\operatorname{Im} \tilde{C}(p)^{\perp} \subset \mathbb{R}[p]^6$, the orthogonal complement of $\operatorname{Im} \tilde{C}(p)$.

LEMMA 3.1. *The module* $\operatorname{Im} \tilde{C}(p)^{\perp}$ *is generated by only one vector*

$$\Lambda(p) = (p_2 p_4(2p_2 p_3 - p_1 p_4 - p_3 p_4),\ p_1 p_3 p_4(p_4 - p_2).\ p_1 p_3 p_4(p_1 - p_3),$$
$$p_1 p_2 p_4(p_4 - p_2),\ p_1 p_2 p_4(p_1 - p_3),\ p_1 p_3(2p_2 p_3 - p_1 p_2 - p_1 p_4)).$$

Proof. Every vector $\Lambda = (\Lambda_1, ..., \Lambda_6) \in \operatorname{Im} \tilde{C}(p)^{\perp}$ belongs to $\operatorname{Ker} \tilde{C}(p)^*$, i.e.,

$$\tilde{C}(p)^*\, \Lambda = \begin{bmatrix} 0 & -p_2 & 0 & p_3 & 0 & 0 \\ 0 & p_1 - p_3 & -p_4 & 0 & p_3 & 0 \\ 0 & p_2 & 0 & -p_1 & p_4 - p_2 & 0 \\ 0 & 0 & p_2 & 0 & -p_3 & 0 \\ p_1 & p_2 & 0 & p_3 & p_4 & 0 \\ 0 & p_1 & p_2 & 0 & p_3 & p_4 \end{bmatrix} \cdot \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \\ \Lambda_5 \\ \Lambda_6 \end{bmatrix} = 0. \tag{21}$$

We solve this system of linear equations. From the first equation in (21) it follows that $\Lambda_2 = p_3 \Lambda_2'$; $\Lambda_4 = p_2 \Lambda_2'$ and from the fourth one that $\Lambda_3 = p_3 \Lambda_3'$, $\Lambda_5 = p_2 \Lambda_3'$ for some $\Lambda_2', \Lambda_3' \in \mathbb{R}[p]$. Next, from the second equation in (21) one obtains that $p_3(p_1 - p_3)\, \Lambda_2' = (p_4 - p_2)\, p_3 \Lambda_3'$ and hence $\Lambda_2' = (p_4 - p_2)\, \Lambda_2''$ and $\Lambda_3' = (p_1 - p_3)\, \Lambda_2''$ for certain $\Lambda_2'' \in \mathbb{R}[p]$. The third equation in (21) is fulfilled automatically. Finally from the last two equations in (21) we have

$$p_1 \Lambda_1 + p_2(p_3 p_4 + p_1 p_4 - 2p_2 p_3)\, \Lambda_2'' = 0,$$
$$p_4 \Lambda_6 + p_3(p_1 p_2 + p_1 p_4 - 2p_2 p_3)\, \Lambda_2'' = 0.$$

This gives $\Lambda_2'' = p_1 p_4 \Lambda'''$, $\Lambda_1 = p_2 p_4(2p_2 p_3 - p_1 p_4 - p_3 p_4)\, \Lambda'''$ and $\Lambda_6 = p_1 p_3(2p_2 p_3 - p_1 p_2 - p_1 p_4)\, \Lambda'''$ for certain $\Lambda''' \in \mathbb{R}[p]$. From this, the assertion of Lemma 3.1 follows.

Because $\operatorname{Im} \tilde{C}(p)^{\perp}$ is one dimensional, any invariant and linear function on $\mathbb{R}[p]^6$ is proportional to $I_1(p, q) = (A(p), q)$. This completes the proof of the point (a) of Proposition 3.1.

(b) Let I/I' be an absolute invariant, where I and I' are relative polynomial invariants with the same weights $\chi = \chi'$. Let $r = dI_1^{\alpha} \prod_{i=1}^4 p_i^{\alpha_i}$ and $r' = d'I_1^{\alpha'} \prod_{i=1}^4 p_i^{\alpha_i'}$ be some monomials in I and I', respectively. Then the equality of weights $\chi = \chi'$ gives $\alpha = \alpha'$, $\alpha_1 + \alpha_3 = \alpha_1' + \alpha_3'$, and $\alpha_2 + \alpha_4 = \alpha_2' + \alpha_4'$ (see (20)). This means that I/I' can be represented as a rational function of $a = p_3 p_1$ and $b = p_4/p_2$. Proposition 3.1 is proved.

Remark 3.1. (a) Besides the rational absolute invariants of $\tilde{\tilde{\mathscr{G}}}_0$ defined in Proposition 3.1 there are the following (nonrational) invariants

$$\operatorname{sign}(p_1 p_2 I_1) \qquad \text{and} \qquad \operatorname{sign}(p_1 p_2 I_2),$$

where

$$I_2 = p_1 p_4 - p_2 p_3. \tag{22}$$

(b) It is not difficult to find the partition of the space j^3Q^2 into orbits of $\tilde{\tilde{\mathscr{G}}}_0$. We shall not do it.

(c) Obviously the formula $I(V) = I(j^3V)$ gives the extension of the invariant I to the whole subspace of singular germs. These extensions of the functions (20) form the invariants of the action of the group of C^∞-orbital equivalences preserving the coordinate axes.

(d) The generator of $\mathbb{Z}_2$ in (16) (change $(x, y) \to (y, x)$) induces the following change in j^3Q^2: $(p, q) \to (p', q')$, where $p' = (p_4, p_3, p_2, p_1)$ and $q' = (q_6, ..., q_1)$. From the formulas (20) and (22) it follows that $I_1' = I_1$ and $I_2' = I_2$.

(e) We choose other generators of the field of absolute invariants, α and β, defined by the formulas

$$a = -\frac{\alpha + 1}{\beta}, \qquad b = -\frac{\alpha}{\beta + 1}. \tag{23}$$

The action of the generator of $\mathbb{Z}_2$ in (16) induces the transformation $(\alpha, \beta) \to (\beta, \alpha)$. For this reason in Fig. 1 we consider only those pairs such that $\alpha \leqslant \beta$.

B. *Action of* $\mathfrak{D}_3$ *on* j^2Q^2

It turns out that on the space

$$\overline{j^2Q^2} = \{V^p = x(p_1 x + p_2 y)\,\partial_x + y(p_3 x + p_4 y)\,\partial_y : p_1 \neq p_3,\ p_2 \neq p_4\}$$

the discrete group $\mathfrak{D}_3 = S(3)$ of permutations of three elements acts. We describe this action by describing two transformations of $\mathbb{R}^2$ which generate the group $S(3)$.

The first transformation is

$$\sigma_1: \quad (x, y) \to (y, x). \tag{24}$$

σ_1 induces the transformation

$$\bar{\sigma}_1: \quad (p_1, p_2, p_3, p_4) \to (p_4, p_3, p_2, p_1) \tag{25}$$

described in Remark 3.1d.

To define the second transformation we note that the vector field V^p has three invariant lines: l_1: $x=0$, l_2: $y=0$, and l_3: $(p_3-p_1)\,x+(p_4-p_2)\,y=0$. Let us change the order of these lines $((1, 2, 3) \to (1, 3, 2))$ and define the change of variables

$$\sigma_2: \quad (x, y) \to \left(x, \frac{p_3-p_1}{p_4-p_2}\,x+y\right) \tag{26}$$

realizing this permutation. This change transforms the vector field V^p to $V^{p'}$, where

$$p' = \bar{\sigma}_2(p) = (p_1 p_4 - p_3 p_2, p_2, 2p_1 p_4 - p_1 p_2 - p_3 p_4, p_4).$$

In terms of the absolute invariants α and β the transformation $\bar{\sigma}_1$ exchanges α and β (see Remark 3.1e) and the transformation $\bar{\sigma}_2$ transforms (α, β) to

$$\bar{\sigma}_2(\alpha, \beta) = (\alpha/\beta, 1/\beta). \tag{27}$$

The above transformations generate the group $S(3) = \{$permutations of l_1, l_2, $l_3\}$. In Section 6 we shall return to this action and shall extend it to all vector fields (15).

C. *Nondegeneracy Conditions*

We define nine submanifolds in $\mathscr{F}$ of codimension 3:

$$Q_i^3 = \{V \in Q^2: p_i(V) = 0\}, \qquad i = 1, 2, 3, 4;$$

$$Q_5^3 = \{V \in Q^2: (p_3 - p_1)(V) = 0\};$$

$$Q_6^3 = \{V \in Q^2: (p_4 - p_2)(V) = 0\};$$

$$Q_7^3 = \{V \in Q^2: I_2(V) = 0\};$$

$$Q_8^3 = \left\{V \in Q^2: I_1(V) = 0, \frac{p_3}{p_1}(V) < \frac{p_4}{p_2}(V) < 0, (p_1 p_2)(V) \neq 0\right\};$$

$$Q_9^3 = \left\{V \in Q^2: I_1(V) = 0, 0 < \frac{p_4}{p_2}(V) < \frac{p_3}{p_1}(V), (p_1 p_2)(V) \neq 0\right\}.$$

Its union $\bigcup Q_i^3$ is invariant with respect to the C^∞-orbital equivalence group $\mathscr{G}$.

DEFINITION 4. Deformation V_μ of the singular vector field V_0 is called *nondegenerate* iff the mapping

$$\mathbb{R}^2 \ni \mu \to V_\mu$$

is transversal at the point $0 \in \mathbb{R}^2$ to the submanifolds Q^2 and Q_i^3, $i = 1, \ldots, 9$. The other deformations we call *degenerate.*

Remark 3.2. The nondegeneracy condition means that $p_i(V_0) \neq 0$, $i = 1, 2, 3, 4$, $p_1(V_0) \neq p_3(V_0)$, $p_2(V_0) \neq p_4(V_0)$, $I_2(V_0) \neq 0$, $I_1(V_0) \neq 0$ if $a(V_0) < b(V_0) < 0$ or $0 < b(V_0) < a(V_0)$ and $\det(\partial(W_1, W_2)/\partial(\mu_1, \mu_2))(0) \neq 0$ (see (6)). All the above conditions are connected with the bifurcations of singular points and periodic orbits of V_μ (see Sects. 4 and 5). In terms of parameters α and β the conditions $a < b < 0$ and $0 < b < a$ mean that (α, β) is contained in one of the domains VI–VIII in Fig. 1.

From Definition 4 and Remark 3.2 point (a) of Theorem 1, degenerate families form a union of codimension one submanifolds, follows.

D. *Normal Forms of Nondegenerate Families*

We finish Section 3 describing the normal forms of the nondegenerate families (6).

PROPOSITION 3.2. *Every nondegenerate family* (6) *is C^∞-orbitally equivalent to the following one*:

$$x(\mu_1 + x + y + \varphi_1)\,\partial_x + y\left(\mu_2 - \frac{\alpha+1}{\beta}\,x - \frac{\alpha}{\beta+1}\,y + \nu R(x+\mu_1, y) + \varphi_2\right)\partial_y, \tag{28}$$

where $\alpha = \alpha_\mu$, $\beta = \beta_\mu$, and (α_0, β_0) are as in Definition 2. *If (α_0, β_0) belongs to the domains* VI–VIII (*Fig.* 1) *then R and ν are as in Definition* 2 *otherwise $\nu = 1$ and R is some quadratic form. $\varphi_{1,2} = O(|\tilde{x}, \tilde{y})|^3)$, where $\tilde{x} = x - x_4$, $\tilde{y} = y - y_4$,* and *(x_4, y_4) is the singular point of V_μ (solution of $W_1 = W_2 = 0$).*

Proof. Let $\mu = 0$. By means of dilations we can reduce $p_i = p_i(V_0)$, $i = 1, 2$, to 1. Therefore, the quadratic part of V_0 is such as in (29) and the nondegeneracy condition means that α, $\beta \neq 0, -1$ and $\alpha + \beta = -1$. Eventually changing the axes we can assume that $\alpha \leqslant \beta$.

Now, we consider $j^3(V_0)$. We shall find $\operatorname{Im} \tilde{C}(p)^\perp \subset \mathbb{R}^6$, where $p = p(V_0)$ and $\tilde{C}(p)$ is defined in (19). Repeating the proof of Lemma 3.1 we see that $\operatorname{Im} \tilde{C}(p)^\perp$ is one dimensional and is generated by the vector $\Lambda(p)$ (due to the nondegeneracy condition). Hence the function $I_1(p, \cdot) = (\Lambda(p), \cdot)$ parametrizes the orbits of the following action of $\mathbb{R}^6$ on $\mathbb{R}^6$: $\omega q = q + \tilde{C}(p)\omega$, $\omega, q \in \mathbb{R}^6$. Standard computations show that for $\beta \neq -2$ the function $I_1(p, \ )$ takes the same values at the points $q = q(V_0)$ and $\tilde{q} = (0, 0, 0, \tilde{\tilde{q}}/\beta, 2\tilde{\tilde{q}}/(\beta+1), \tilde{\tilde{q}}/(\beta+2))$, where $\tilde{\tilde{q}} = I_1(p, q)\, \beta^2(\beta+1)^2/(\alpha+\beta+1)^2$. If $\beta = -2$ then $I_1(p, q) = I_1(p, \tilde{q})$, where $\tilde{q} = (0, 0, 0, \tilde{\tilde{q}}, 0, 0)$, $\tilde{\tilde{q}} = I_1(p, q)/\alpha(\alpha+\beta+1)$. Since $I_1(p, q) \neq 0$ for (α, β) from the domains VI–VIII we can make $\tilde{\tilde{q}} = \pm 1$ (using dilations preserving $p_1 = p_2 = 1$). In other cases $\tilde{\tilde{q}} = 0, \pm 1$. The described above action of $\mathbb{R}^6$ on $\mathbb{R}^6$ corresponds exactly to the equivalence relation (13). So Proposition 3.2 in the case $\mu = 0$ is proved.

Let $\mu \neq 0$. (Recall that p_i may depend on μ.) From the above and from Implicit Function Theorem it follows that for small $|\mu|$ the quadratic and cubic terms of V_μ can be reduced to

$$x(x+y)\,\partial_x + y(-(\alpha+1)\,x/\beta - \alpha y/(\beta+1) + vR(x, y))\,\partial_y.$$

Let (x_4, y_4) be the singular point of V_μ, the solution of $W_1 = W_2 = 0$. We write

$$W_i = a_i + b_i x + c_i y + d_i x^2 + e_i xy + f_i y^2 + \varphi_i, \qquad i = 1, 2,$$

where $\varphi_i = O(|(\tilde{x}, \tilde{y})|^3)$ and $a_i, \dots, f_i$ depend on μ. Since the first terms (of order $\leqslant 2$) of the expansion of φ_i at 0 are small, using the above remark, we can transform V_μ to another one such that $b_1 = c_1 = 1$, $d_1 = e_1 = f_1 = 0$, and $d_2 x^2 + e_2 xy + f_2 y^2 = vR(x, y)$. Obviously W_2 can be rewritten as $a_2 + b_2 x + c_2 y + vR(x + a_1, y) + \varphi_2)$. Finally the nondegeneracy condition $\det(\partial(W_1, W_2)/\partial(\mu_1, \mu_2))(0) \neq 0$ means that we can choose a_1 and a_2 as new parameters. Proposition 3.2 is complete.

4. Singular Points and Their Bifurcations

In this section we analyze the topological character of the singular points of the vector field (28).

A. *The Case* $\mu = 0$

If $\mu = 0$, then in a neighbourhood of $0 \in \mathbb{R}^2$ the vector field V_0 has only one singular point 0 and its topological type is completely determined by the quadratic part of V_0, $j^2 V_0$.

Using the polar blowing-up [17, 41] we get the following system

$$\begin{aligned} \dot{r} &= r\left[\cos^2\theta(\cos\theta+\sin\theta)+\sin^2\theta\left(-\frac{\alpha+1}{\beta}\cos\theta-\frac{\alpha}{\beta+1}\sin\theta\right)\right], \\ \dot{\theta} &= -(\alpha+\beta+1)\cos\theta\sin\theta\left(\frac{1}{\beta}\cos\theta+\frac{1}{\beta+1}\sin\theta\right). \end{aligned} \tag{29}$$

LEMMA 4.1. *The vector field* (29) *has following singular points at the circle* $r=0$,

$$\begin{aligned} &Z_1=(0,0), && dV(Z_1)=\begin{bmatrix}1 & 0\\ 0 & -(\alpha+\beta+1)/\beta\end{bmatrix}; \\ &Z_2=(0,\pi), && dV(Z_2)=-dV(Z_1) \\ &Z_3=(0,\pi/2), && dV(Z_3)=\begin{bmatrix}-\alpha/(\beta+1) & 0\\ 0 & (\alpha+\beta+1)/(\beta+1)\end{bmatrix}; \\ &Z_4=(0,-\pi/2), && dV(Z_4)=-dV(Z_3); \\ &Z_5=\left(0,-\arctan\frac{\beta+1}{\beta}\right)=(0,\tilde{\theta}), \end{aligned} \tag{30}$$

$$dV(Z_5)=\cos^3\tilde{\theta}\begin{bmatrix}-((\beta+1)^2+1)/\beta & 0\\ 0 & (\alpha+\beta+1)(\beta^2+\beta+1)/\beta^3\end{bmatrix};$$

$$Z_6=(0,\pi-\tilde{\theta}), \qquad dV(Z_6)=-dV(Z_5).$$

These singular points of the system (29) correspond to three invariant lines of the vector field j^2V_0. From (30) one can easily draw the phase portraits of j^2V_0 (Fig. 2). Analogous pictures are in the papers of Dumotier [17] and Takens [41], in [13] and in [2].

From Fig. 2 it is seen that there are 5 (topological) equivalence classes of V_0 (of the form (8) or (28)). The cases I and II (Fig. 1) are the same. The same holds for the cases III, IV, and V. The next diffeentiation reveals in the bifurcations of singular points for $\mu\neq 0$.

B. *The Case* $\mu\neq 0$

It is natural to expect that for small $|\mu|$ the singular points of V_μ and their eigenvalues are close to that one computed for the following vector field

$$j^2V_\mu=x(\mu_1+x+y)\,\partial_x+y(\mu_2+ax+by)\,\partial_y, \tag{31}$$

where $a=-(\alpha+1)/\beta$ and $b=-\alpha/(\beta+1)$.

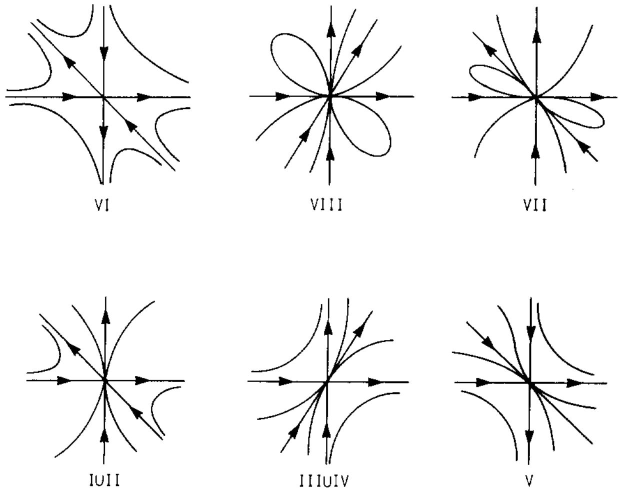

FIGURE 2

LEMMA 4.2. *The vector field* (31) *has following singular points* $P_i = (x_i, y_i)$ *and linear parts at them*:

$$
\begin{aligned}
&P_1 = (0, 0), && dV(P_1) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}; \\
&P_2 = (0, -\mu_2/b), && dV(P_2) = \begin{bmatrix} \mu_1 - \mu_2/b & 0 \\ -a\mu_2/b & -\mu_2 \end{bmatrix}; \\
&P_3 = (-\mu_1, 0), && dV(P_3) = \begin{bmatrix} -\mu_1 & -\mu_1 \\ 0 & \mu_2 - a\mu_1 \end{bmatrix}; \\
&P_4 = \left(\frac{b\mu_1 - \mu_2}{a-b}, \frac{\mu_2 - a\mu_1}{a-b} \right), && dV(P_4) = \begin{bmatrix} x_4 & y_4 \\ ay_4 & by_4 \end{bmatrix}.
\end{aligned}
\tag{32}
$$

We see that if $\mu_1 \neq 0$, $\mu_2 \neq 0$, $\mu_2 \neq a\mu_1$, and $\mu_2 \neq b\mu_1$ then the points P_1, P_2, and P_3 are hyperbolic. The lines $\mu_1 = 0$, $\mu_2 = 0$, $\beta\mu_2 + (\alpha + 1)\mu_1 = 0$, and $(\beta + 1)\mu_2 + \alpha\mu_1 = 0$ are bifurcational. If μ belongs to one of them then two of singular points of $j^2 V_\mu$ are equal. This point is of the saddle node type. This follows from the analysis of the second order terms in the expansion of $j^2 V_\mu$ at this point. In the family (28) the bifurcational curves are little changed (the difference is of order $|\mu|^2$). The topological types of corresponding quasihyperbolic points are the same as for the vector field (31). At this moment we can distinguish those cases which could not be differentiated when $\mu=0$ (because for $\alpha=-1$ or for $\beta=-1$ two of the above bifurcational lines coincide).

C. *Investigation of* P_4

If P_4 is of the saddle node type then the considerations are analogous to those which concern P_1, P_2, and P_3. The interesting is the case when P_4 is a focus which loses stability. We look when it is possible. We have $\operatorname{tr} dV(P_4)=x_4+by_4=0$ and $\det dV(P_4)=(b-a)\,x_4y_4>0$. Therefore $-b(b-a)>0$. If $b<0$ then $a<0$ and this case corresponds to $0<\alpha\leqslant\beta$ (domain VI at Fig. 1). If $b=-\alpha/(\beta+1)>0$ then $a>b$. In the last case $\alpha<0$ (because $\alpha\leqslant\beta$) and then $\beta>-1$. The condition $a>b$ states that $-(\alpha+\beta+1)/\beta(\beta+1)>0$. Hence either $\beta>0$, $\alpha+\beta+1<0$ (domain VII) or $-1<\beta<0$, $\alpha+\beta>-1$ (domain VIII). Generally, P_4 changes stability iff $\mu_2=b(a-1)\,\mu_1/(b-1)=-\alpha\mu_1/\beta$ and (α,β) belongs to one of the domains VI–VIII. The problem of stability of P_4 at bifurcational curve and creating limit cycles from P_4 we leave to the next section.

In what follows we use another notation for the domains VI, VII, and VIII. Let us note that they differ in the range of values of $\alpha+\beta$: $\alpha+\beta>0$ for the case VI, $-1<\alpha+\beta<0$ for the case VIII and $\alpha+\beta<-1$ for the case VII. The corresponding cases we shall denote by the corresponding intervals of values of $\alpha+\beta$.

Now, let us look at signs of x_4 and y_4. Since $x_4=-by_4$ we have either $x_4<0$, $y_4<0$ or $x_4>0$, $y_4>0$ $(b<0)$ and either $x_4<0$, $y_4>0$ or $y_4<0<x_4$ $(b>0)$. We focus our attention on the case $y_4>0$. Then one can easily check that

$$\mu_1<0<\mu_2, \qquad x_4>0 \quad \text{if} \quad \alpha+\beta>0,$$

$$\mu_1<0<\mu_2, \qquad x_4<0 \quad \text{if} \quad -1<\alpha+\beta<0,$$

and (33)

$$\mu_1,\mu_2>0, \qquad x_4<0 \quad \text{if} \quad \alpha+\beta<-1.$$

The other case $(y_4<0)$ is considered analogously. We want to work in the first quarter $x,y\geqslant 0$ of $\mathbb{R}^2$. For this reason we use the transformation $x\to -x$ in the case $\alpha+\beta<0$.

D. *Bifurcations of Singular Points in the Quadrant* $x\geqslant 0$, $y\geqslant 0$

In this subsection we shall find a correct partition of the space of the families of the form (4) restricted to the quadrant $x\geqslant 0$, $y\geqslant 0$ into components corresponding to different topological equivalence classes. Let us note that the transformations $x\to -x$ and $y\to -y$ are not allowed in our situation. As in [21] and [25] we allow the reversion of time.

Using the extension of coordinates and changing eventually the direction of time one can obtain the following system

$$A_{\pm}: \begin{cases} \dot{x} = x(\mu_1 + x \pm by) \\ \dot{y} = y(\mu_2 + cx \pm y). \end{cases}$$

(a) At the beginning we observe that the change $(x, y, t) \to (y, x, \pm t)$ puts the family $A_{\pm}$ to $A_{\pm}$ and also $(b, c) \to (c, b)$. Therefore we can restrict our attention to the domain $c \leqslant b$.

(b) Let $\mu = 0$. Then the blowing-up construction gives the following system

$$\dot{r} = r[\cos^3\theta \pm b\cos^2\theta \sin\theta + c\cos\theta\sin^2\theta \pm \sin^3\theta]$$
$$\tilde{\theta} = \cos\theta\sin\theta[(c-1)\cos\theta \mp (b-1)\sin\theta].$$

In the domain $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$ this system has the following singular points and corresponding matrices of linear parts

$$(0, 0), \qquad \begin{bmatrix} 1 & 0 \\ 0 & c-1 \end{bmatrix},$$

$$(0, \pi/2), \qquad \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm(b-1) \end{bmatrix},$$

$$(0, \tilde{\theta}), \qquad \pm\frac{(b-1)^2 + (c-1)^2}{(c-1)^2}\sin^3\tilde{\theta} \begin{bmatrix} \dfrac{bc-1}{c-1} & 0 \\ 0 & -(b-1) \end{bmatrix},$$

where $\tilde{\theta} = \pm\arctan(c-1)/(b-1)$ if $\pm(c-1)/(b-1) > 0$. From this the types of the phase portraits at $\mu = 0$ follow (see Fig. 3).

Let us note that the half-line $b = 1$, $c \leqslant 1$ in the A_- case is singular. (In the case A_-, $bc > 1$ and $c < 1$ there is a parabolic sector but it is nonessential from the topological point of view.)

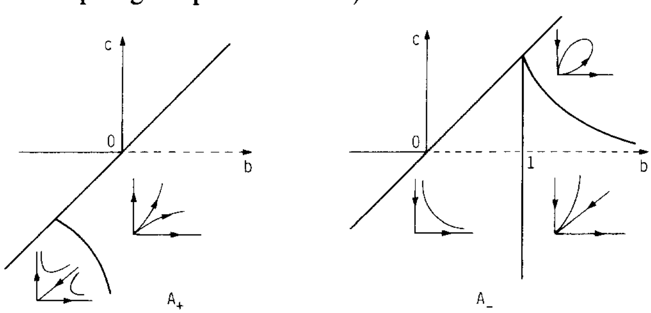

FIGURE 3

(c) Let $\mu \neq 0$. Then the system $A_{\pm}$ has four singular points

$$(0,0), \quad \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

$$(0, \mp\mu_2), \quad \begin{bmatrix} \mu_1 - b\mu_2 & 0 \\ * & -\mu_2 \end{bmatrix}, \quad \mp\mu_2 > 0,$$

$$(-\mu_1, 0), \quad \begin{bmatrix} -\mu_1 & * \\ 0 & \mu_2 - c\mu_1 \end{bmatrix}, \quad \mu_1 < 0,$$

$$(\tilde{x}, \tilde{y}) = \left(\frac{\mu_1 - b\mu_2}{bc-1}, \mp\frac{\mu_2 - c\mu_1}{bc-1}\right),$$

$$L = \begin{bmatrix} \tilde{x} & \pm b\tilde{x} \\ c\tilde{y} & \pm\tilde{y} \end{bmatrix}, \quad \tilde{x} > 0,\ \tilde{y} > 0,\ bc \neq 1.$$

It is clear that the lines K: $\mu_1 = 0$, L: $\mu_2 = 0$, M: $\mu_1 = b/\mu_2$, $\pm\mu_2 \geqslant 0$, and N: $\mu_2 = c\mu_1$, $\mu_1 \leqslant 0$ at μ-plane are bifurcational. Therefore the curves $b = 0$, $c = 0$, and $bc = 1$ at bc-plane are singular. (At the curves K, L, M, N the system $A_{\pm}$ undergoes saddle node bifurcations.)

There are no foci losing the stability in the case A_+. It is possible in the case A_-. Then $\tilde{x} = \tilde{y} > 0$ (or Q: $(b-1)\,\mu_2 = (1-c)\,\mu_1$, $(b-1)\,\mu_1 > 0$) and $\det L = (bc-1)\,\tilde{x}^2 > 0$. This means that the curve $c = 1$ is singular. (The conditions $(b-1)/(1-c) \neq b$ and $(1-c)/(b-1) \neq c$ hold because $bc \neq 1$.) The complete bc-diagrams are presented at Fig. 4.

(d) If one forbids the change $t \to -t$ then one obtains 2 copies of A_+ (Fig. 4) and one copy of the bc-plane cut along the curves $b = 0, 1$, $c = 0, 1$ and $bc = 1$. The number of corresponding components is 22.

(e) If one take into consideration additionally the terms of 3rd order

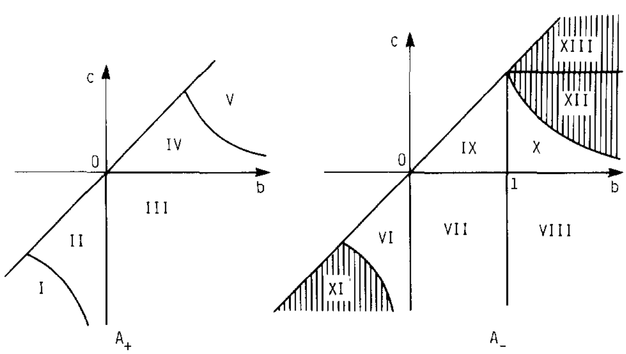

FIGURE 4

in (3) then the "difficult components" XI, XII, and XIII (Fig. 4) must be counted double. The whole number of components is then equal to 26. (Theorem 1c asserts that the "amplitude systems" are divided into only 11 classes.)

(f) In Guckenheimer and Holmes [25], the lines $b=1$ and $c=1$ in the case A_- are missed and the domains III and VII, VIII are counted double. In Gavrilov [21] the domain V is showed wrong and the half-line $b=1$, $c \leqslant 1$ is missed.

(g) One can easily check that the domains XI, XII, and XIII at Fig. 4 correspond to the domains VI, VII, and VIII at Fig. 1, respectively.

(h) The bifurcational diagrams of the families $A_\pm$ consists of:

(α) 4 curves K, L, M, and N in the cases I–X.

(β) 5 curves K, L, M, N, and Q in the cases XI–XIII (see the point (c)).

If one take into account the third order terms in the family (3) then the bifurcational diagram contains additionally the curve $S: (b-1)\,\mu_2 = (1-c)\,\mu_1 + \kappa(\mu_1, \varepsilon(\mu_1, \varepsilon)$, $(b-1)\,\mu_1 > 0$, where κ is certain function satisfying $\kappa(\mu_1, \varepsilon) = o(\mu_1)$ as $(\mu_1/\varepsilon) + \varepsilon \to 0$ (in the cases XI–XIII). For μ belonging to the domain between Q and S the vector field V_μ has a unique limit cycle. We omit the derivation of the asymptotic formula for the function κ (analogous to (11) and (12)).

5. Bifurcations of Limit Cycles

Let us consider the system (28) with (α, β) belonging to one of the domains VI, VII, or VIII and with $\nu = 1$. (The case $\nu = -1$ differes only in direction of time.) When parameter μ passes the curve $Q: \mu_2 \sim -\alpha\mu_1/\beta$, where the singular point P_4 losses the stability a limit cycle appears. The cycle should increase [1, 3]. The goal of this and next section is to investigate the cycle. Hence the rest of this work is devoted to the proof of Theorem 2.c.

A. *Reduction to the Perturbation of Conservation System*

The main idea of the proof of Theorem 2c is the same as in other problems of similar nature [1, 3, 7, 8, 13, 25, 30, 40, 45]. The system (28) is the small perturbation of the system having the first integral. To assure we use the following change of variables. First, we change the sign of x in the case $\alpha + \beta < 0$. Then the point P_4 belongs to the domain $x \geqslant 0$, $y \geqslant 0$ (see the Sect. 4C). The subsequent changes of variables are different with respect to different situation. The general form of the change is the following:

$$x' = x/\gamma_2, \qquad y' = y/\gamma_2, \qquad t' = \gamma_2 t. \tag{34}$$

Now, we specify the constant γ_2. If Γ is the cycle under consideration then we define

$$x(\Gamma) = \max\{x: \exists y (x, y) \in \Gamma\}.$$

We put

$$\gamma_2 = \begin{cases} x(\Gamma) & \text{if } \ |\mu_1/x(\Gamma)| \ll 1 \quad (\Gamma \text{ is big}) \\ |\mu_1| & \text{otherwise.} \end{cases}$$

Let us define

$$\lambda = \mu_1/x(\Gamma)$$

and define the function $s(\lambda)$ as in Table I in the first case and put

$$\lambda = \operatorname{sign} \mu_1$$

and $s(1) = 1$ in the second case. The vector field (28) transforms to the following one:

$$V_{\lambda,\gamma} = V_{\lambda,0} - \gamma_1 V_1 + \gamma_2 V_2 + V_3, \tag{35}$$

where

$$V_{\lambda,0} = x(\lambda + \eta x + y)\,\partial_x + y\left(-\frac{\alpha}{\beta}\lambda - \eta\frac{\alpha+1}{\beta}x - \frac{\alpha}{\beta+1}y\right)\partial_y, \tag{36}$$

$$V_1 = \frac{s(|\lambda|)}{\beta}\, y\partial_y, \qquad V_2 = yR(\lambda + \eta x, y)\,\partial_y, \tag{37}$$

$$V_3 = O(|\gamma|^2\,|(\tilde{x}, \tilde{y})|^3),$$

where $\eta = \operatorname{sign}(\alpha + \beta)$, $(\tilde{x}, \tilde{y}) = (x - x_4, y - y_4)$, $\gamma = (\gamma_1, \gamma_2)$ and

$$\gamma_1 = -\beta\left(\mu_2 + \frac{\alpha}{\beta}\mu_1\right)\Big/\gamma_2 s(|\lambda|). \tag{38}$$

Since $\mu \in \mathcal{O}_\varepsilon$ and $\Gamma \subset \mathcal{U}_\varepsilon$ (see (10)) one has $0 < \gamma_2 < \text{const.}\ \varepsilon$.

In the sequel we assume that $V_3 \equiv 0$. This assumption we make for the sake of simplicity. From the proof it will be seen that all assertions below remains true for $V_3 \neq 0$ too.

It turns out that the vector field $V_{\lambda,0}$ is conservative.

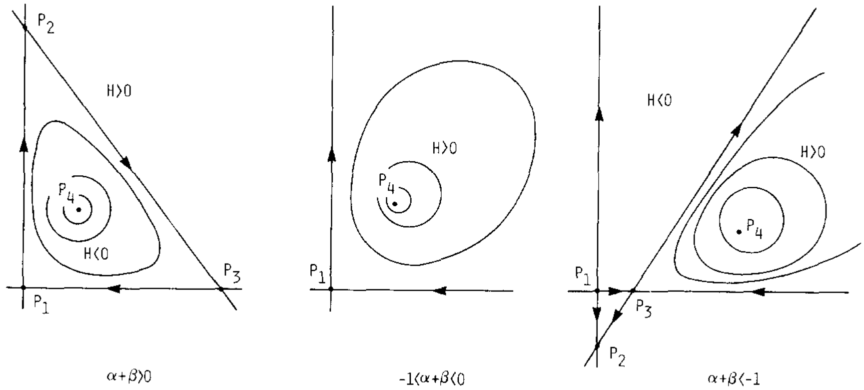

FIGURE 5

LEMMA 5.1 (Bautin [4]). *The vector field $V_{\lambda,0}$ has the following first integral*:

$$H(x, y) = H_\lambda(x, y) = x^\alpha y^\beta \left(\frac{y}{\beta+1} + \frac{\lambda+\eta x}{\beta}\right). \tag{39}$$

Proof. The vector field $x^{\alpha-1}y^{\beta-1}V_{\lambda,0}$ is a Hamiltonian vector field with H as a Hamiltonian (see Fig. 5).

From (35)–(37) we see that the vector field $V_{\lambda,\gamma}$ is a small perturbation of the vector field $V_{\lambda,0}$. As in other similar situations we expect that in the perturbed system there appears a limit cycle Γ close to certain curve $H^{-1}(h)$ and it changes with the parameter γ.

The further part of the proof we divide into four steps: Γ is small, Γ is close to the separatrice contour $\Delta P_1P_2P_3$, diameter of Γ is much greater than $|(x_4, y_4)|$ and an intermediate case. We start with the first one.

B. *Γ is Small* ($|\gamma_1/\gamma_2| \ll 1$)

Here all the subcases (different $\alpha+\beta$) are considered together and we use the change (34) with $\gamma_2 = |\mu_1|$. Instead of the vector field $V_{\lambda,\gamma}$ we analyze the vector field $\tilde{V}_{\lambda,\gamma} = x^{\alpha-1}y^{\beta-1}V_{\lambda,\gamma}$ which we split as follows

$$\tilde{V}_{\lambda,\gamma} = \tilde{V}_0 - \gamma_1\tilde{V}_1 + \gamma_2\tilde{V}_2,$$

where

$$\begin{aligned}\tilde{V}_0 &= x^{\alpha-1}y^{\beta-1}V_{\lambda,0} - \gamma_1\beta^{-1}x^{\alpha-1}(-\lambda-\eta x)^\beta\,\partial_y\\ &\quad + \gamma_2 x^{\alpha-1}(-\lambda-\eta x)^\beta R(\lambda+\eta x, -\lambda-\eta x)\,\partial_y,\\ \tilde{V}_1 &= \beta^{-1}x^{\alpha-1}[y^\beta - (-\lambda-\eta x)^\beta]\,\partial_y,\end{aligned}$$

and

$$\tilde{V}_2 = x^{\alpha-1}[y^{\beta}R(\lambda+\eta x, y) - (-\lambda-\eta x)^{\beta} R(\lambda+\eta x, -\lambda-\eta x)]\,\partial_y.$$

We have $\tilde{V}_i(P_4)=0$, $i=0,1,2$. $\tilde{V}_0$ is a Hamiltonian vector field with a Hamiltonian $\tilde{H}$. In a small neighbourhood of $P_4=(x_4, -\lambda-\eta x_4)$ $\tilde{H}-\tilde{H}(P_4)$ is approximately a quadratic form of $(\tilde{x}, \tilde{y})$, $\tilde{V}_1$ is of order $O(|(\tilde{x}, \tilde{y})|)$ and $\tilde{V}_2$ is of order $O(|(\tilde{x}, \tilde{y})|^3)$. The latter follows from the identity $(\partial/\partial y)\, y^{\beta}R(\lambda+\eta x, y) = y^{\beta-1}(\lambda+\eta x+y)^2$ (see (9)).

Therefore we have the situation as in the Hopf bifurcation [1, 3]. The vector field $\tilde{V}_{\lambda,\gamma}$ (and $V_{\lambda,\gamma}$ too) has an unstable limit cycle Γ_h close to the curve $H^{-1}(h)$, which is close to the ellipse $kx^2+lxy+my^2=r^2$ iff $\gamma_1 = \text{const.}\, \gamma_2 r^2(1+O(|r|+|\gamma|))$. We obtain the last assertion integrating the increment of the function $\tilde{H}$ along the integral curve of $\tilde{V}_{\lambda,\gamma}$. As a consequence we get that there exists $\delta_0>0$ such that for $0 \leqslant |\gamma_1/\gamma_2| < \delta_0$ and $\gamma_2<\delta_0$, Theorem 2c holds. Here $h-h_{\min} < \text{const.}\, \delta_0$, where $h_{\min}=H(P_4)$.

C. *Γ is Close to $\Delta P_1P_2P_3$* $(\alpha+\beta>0,\ -\lambda=\eta=1,\ -1\leqslant h<0)$

First, we give the construction of certain functions called a "succeeding function" which will be useful in the sequel. Let

$$l_1=\left\{\left(x, \frac{\beta+1}{\alpha}\,x\right): 0<x<x_4\right\}, \qquad l_2=\left\{\left(x, \frac{\beta+1}{\alpha}\,x\right): x_4<x\right\}$$

be two intervals parametrized by H (here $P_4=(\alpha+\beta+1)^{-1}\,(\alpha, \beta+1)$). We define two mappings from l_1 to l_2. Let $P\in l_1$, $H(P)=h$. Let $\Gamma_h^{(+)}=\{P(t)$: $t>0\}$ (resp. $\Gamma_h^{(-)}=\{P(t)$: $t<0\}$) be the positive (resp. negative) trajectory of $V_{\lambda,\gamma}$ starting at the point P. Let $P^{(+)}=P(t^{(+)})$ (resp. $P^{(-)}=P(t^{(-)})$) be the point of first intersection of $\Gamma_h^{(+)}$ (resp. $\Gamma_h^{(-)}$) with l_2.

DEFINITION 5. The function

$$\eta(h)=H(P^{(+)})-H(P^{(-)})$$

is called a succeeding function. The domain of definiteness of the function η is $(h_{\min}+\text{const.}\,\delta_0, h_{\max})$ for a certain constant $h_{\max}$, the value of which will follow from the context.

The next lemma follows from standard theorems on differential equations.

LEMMA 5.2. *Let $|\gamma_1/\gamma_2|>\delta_0$ and $h-h_{\min}>\text{const.}\,\delta_0$. Then for sufficiently small $|\gamma|$ is*

(a) *the function η is well defined and smooth in γ and h,*

(b) $\Gamma_h = \{P(t)\colon t \in [t^{(-)}, t^{(+)}]\}$ *is a periodic orbit iff* $\eta(h) = 0$, *and*

(c) *this orbit is repelling (contracting) iff* $\eta'(h) > 0$ (*resp.* $\eta'(h) < 0$).

In the next lemma we give the integral formulas for η and η' (see [1, Sect. 13]).

LEMMA 5.3. (a) *We have*

$$\begin{aligned}\eta(h) &= \int_{t^{(-)}}^{t^{(+)}} \frac{d}{dt} H(P(t))\, dt \\ &= \int_{\Gamma_h} x^{\alpha-1} y^{\beta} \left(\frac{-\gamma_1 s(|\lambda|)}{\beta} + \gamma_2 R(\lambda + \eta x, \eta) \right) dx \\ &= \int_{U_h} x^{\alpha-1} y^{\beta-1} (-\gamma_1 s(|\lambda|) + \gamma_2 z^2)\, dx\, dy, \end{aligned} \tag{40}$$

where U_h *is the domain bounded by* Γ_h (*if* Γ_h *is closed*) *and* $z = \lambda + \eta x + y$.

(b) *If* Γ_h *is a closed trajectory then*

$$\begin{aligned}\eta'(h) &= A(\gamma, h) \int_{t^{(-)}}^{t^{(+)}} (x^{1-\alpha} y^{1-\beta} \operatorname{div}(x^{\alpha-1} y^{\beta-1} V_{\lambda,\gamma}))(P(t))\, dt \\ &= A(\gamma, h) \oint_{\Gamma_h} \frac{-\gamma_1 s(|\lambda|) + \gamma_2 z^2}{xz}\, dx, \end{aligned} \tag{41}$$

where $A(0, h) = 1$.

Proof. The formula (40) is obvious because $dx = xz\, dt$ (*see* (35)–(37)). To prove the formula (41) we assume that $\eta(h) = 0$ and choose the trajectory $P(\tau)$ of the vector field $\tilde{V}_{\lambda,\gamma} = x^{\alpha-1} y^{\beta-1} V_{\lambda,\gamma}$ crossing the point P. We have $dt = x^{\alpha-1} y^{\beta-1}\, d\tau$. Let $\tilde{H}$ be a smooth function defined in a neighbourhood of Γ_h such that $\Gamma_h = \{\tilde{H} = h\}$ and $\tilde{V}_{\lambda,\gamma}|_{\Gamma_h}$ is a Hamiltonian vector field with $\tilde{H}$ as the Hamiltonian. We choose $\tilde{H}$ such that $\tilde{H} = H$ for $\gamma = 0$. Consider the function

$$\rho(\tilde{h}) = \oint_{\tilde{H} = \tilde{h}} \tilde{V}^{\perp}_{\lambda,\gamma}(P(\tilde{\tau}))\, d\tilde{\tau} = \iint_{\tilde{H} < \tilde{h}} \operatorname{div} \tilde{V}_{\lambda,\gamma}\, d\tilde{\tau}\, d\tilde{H},$$

where $\tilde{V}^{\perp}_{\lambda},\gamma$ is the normal to $\{\tilde{H} = \tilde{h}\}$ component of $\tilde{V}_{\lambda,\gamma}$ and $P(\tilde{\tau})$ is the trajectory of the Hamiltonian vector field generated by $\tilde{H}$. Then $\rho(h) = 0$ and $\rho'(h) = \int \operatorname{div} \tilde{V}_{\lambda,\gamma}(P(\tau))\, d\tau$. It is easily to see that $\rho(h)/\eta(h) \to 1$ as $|\gamma| \to 0$. From this the convergence $\rho'(h)/\eta'(h) \to 1$ as $|\gamma| \to 0$ and the formula (41) follow.

We pass to the proof of Theorem 2c in the case when Γ is close to $\Delta P_1P_2P_3$. By Lemma 5.2 it is enough to show that if $\eta(h)=0$ then $\eta'(h)>0$. In fact $\eta'(h)\to\infty$ as $h\to 0$. From this the uniqueness of the limit cycle in this domain follows. By (40),

$$\eta(h)=\oint_{H=h} x^{\alpha-1}y^{\beta}(-\gamma_1\beta^{-1}+\gamma_2R(x-1,y))\,dx+o(|\gamma|).$$

This equality is a consequence of the fact that the curve Γ_h is close to the curve $\{H=h\}$ (difference is of order $|\gamma|$). If $h\to 0$ and $|\gamma|\to 0$ then the integral curve tends to $\Delta P_1P_2P_3$ and one needs to compare with 0 the expression

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta}\,[-\gamma_1/\beta+\gamma_2(1-x)^2/\beta^2(\beta+2)]\,dx\\ &\quad=-\frac{\gamma_1}{\beta}\,B(\alpha,\beta+1)+\frac{\gamma_2}{\beta^2(\beta+2)}\,B(\alpha,\beta+3)\\ &\quad=\frac{1}{\beta}\,\frac{\Gamma(\alpha)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\left(-\gamma_1+\gamma_2\,\frac{\beta+1}{\beta(\alpha+\beta+1)(\alpha+\beta+2)}\right),\end{aligned}$$

where $B(\alpha,\beta)=\Gamma(\alpha)\,\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Betha-function [5]. Hence $\eta=0$ iff

$$\gamma_1=\gamma_2(\beta+1)/\beta(\alpha+\beta+1)(\alpha+\beta+2)+o(|\gamma|).$$

For the family (28) (or (8)) this formula is equivalent to (11) (see (38)).

Now, we show that $\eta'(h)\to+\infty$. We cannot use the formula (41) because the integrals in (41) tend to infinity. However, there is a very simple criterion for this (see [18]).

Let $\lambda_i>0$ and $-\mu_i<0$ be eigenvalues of the linear parts of the vector field $V_{\lambda,\gamma}$ at the saddle points P_i, $i=1,2,3$. Then

$$\text{if } T=\prod_{i=1}^{3}\frac{\lambda_i}{\mu_i}>1 \text{ then } \eta'(h)\to+\infty \text{ as } h\to 0.$$

Obviously if $\gamma=0$ then $T=0$. The expansion of $dV(P_i)$ of the first order with respect to $|\gamma|$ are the following:

$$dV(P_1)=\begin{bmatrix}-1 & 0\\ 0 & \frac{\alpha}{\beta}\left(1-\frac{\gamma_1}{\alpha}+\frac{\gamma_2}{\alpha}\right)\end{bmatrix},$$

$$dV(P_2) = \begin{bmatrix} \frac{1}{\beta}\left(1 - \frac{\beta+1}{\alpha}\gamma_1 + \frac{\beta+1}{\alpha\beta(\beta+2)}\gamma_2\right. & 0 \\ * & -\frac{\alpha}{\beta}\left(1 - \frac{\gamma_1}{\alpha} - \frac{\gamma_2}{\alpha\beta(\beta+2)}\right) \end{bmatrix},$$

$$dV(P_3) = \begin{bmatrix} 1 & * \\ 0 & -\frac{1}{\beta}(1+\gamma_1) \end{bmatrix}.$$

From this one easily finds

$$\begin{aligned} T &= 1 - \frac{1}{\alpha}\left[\frac{\beta+1}{\beta}\gamma_2 - (\alpha+\beta+1)\gamma_1\right] + o(|\gamma|) \\ &= 1 + \frac{\beta+1}{\alpha\beta}\left(1 - \frac{1}{\alpha+\beta+2}\right)\gamma_2 + o(|\gamma|) > 1 \end{aligned}$$

for $\gamma_1 = (\beta+1)\gamma_2/\beta(\alpha+\beta+1)(\alpha+\beta+2) + o(|\gamma|) > 0$.

This ends the proof of Theorem 2c in the case when Γ is close to the separatrice contour.

D. *Γ is Large* ($\alpha+\beta<0$, $|\lambda| \ll 1$)

We recall that $\gamma_2 = x(\Gamma) = \max\{x: \exists y (x, y) \in \Gamma\}$ measures the magnitude of the cycle Γ in the system (28) or (8) and $\lambda = \mu_1/\gamma_2$ and $\gamma_1 = -\beta(\mu_2 + \alpha\mu_1/\beta)/s(|\lambda|)\gamma_2$, where $s(\lambda)$ equals

$$\begin{array}{lll} \lambda/\ln(1/\lambda) & \text{or} & \lambda^{-\alpha-\beta} \quad \text{or} \\ \lambda^2\ln(1/\lambda) & \text{or} & \lambda^2. \end{array} \tag{42}$$

The investigation of the limit cycle in this situation is rather difficult because we must control the trajectories of the vector field $V_{\lambda,\gamma}$ (35) in large ($O(1)$) and small ($O(\lambda)$) domains.

At the beginning we define a new "succeeding function" $\eta(\gamma, \lambda)$ with arguments γ and λ. Denote by l the interval with ends P_1 and P_4 (Fig. 3). We parametrize it by H. Let $\Gamma^{(+)}$ ($\Gamma^{(-)}$) be positive (negative) trajectory of the vector field $V_{\lambda,\gamma}$ crossing P_5. $P_5 = (1, 1-\lambda)$ is the point on the line $x=1$ satisfying $\dot{x}(P_5) = 0$. Let $P^{(+)}(P^{(-)})$ be the point of the first intersection of $\Gamma^{(+)}(\Gamma^{(-)})$ with l.

DEFINITION 6. The function

$$\eta(\gamma, \lambda) = \Delta H = H(P^{(+)}) - H(P^{(-)})$$

is called succeeding function.

We denote by $\Gamma = \{P(t): t \in \mathbb{R}\}$ the trajectory of $V_{\lambda,\gamma}$ crossing P_5 and $\Gamma_0 = \{H(x, y) = H(P_5)\}$. The next lemma is analogous to Lemma 5.2 is standard.

LEMMA 5.4. *Let $|\gamma|$ be sufficiently small. Then*

(a) *the function $\eta(\gamma, \lambda)$ is well defined and smooth,*

(b) *Γ forms a closed orbit* iff $\eta = 0$, *and*

(c) *in this case Γ is repelling (contracting)* iff

$$\left.\frac{d\eta}{d\gamma_2}\right|_{\eta=0} = \left.\left(\frac{-\lambda}{\gamma_2}\frac{\partial\eta}{\partial\lambda} + \frac{\gamma_1}{\gamma_2}\left(\frac{s'(|\lambda|)\,|\lambda|}{s(|\lambda|)} - 1\right)\frac{\partial\eta}{\partial\gamma_1} + \frac{\partial\eta}{\partial\gamma_2}\right)\right|_{\eta=0} > 0 \qquad (43)$$

(resp. $d\eta/d\gamma_2\,|_{\eta=0} < 0$).

The rest of this subsection is devoted to the proof of the following proposition from which (by Lemma 5.4) Theorem 2c in the considered case follows.

PROPOSITION 5.1. *There exists $\delta > 0$ and a germ of smooth function $g(\lambda, \gamma_2)$, $|\lambda|, \gamma_2 < \delta$, such that*

(a) *$\gamma_1 = g(\lambda, \gamma_2)$ is the solution of the equation $\eta = 0$,*

(b) *$d\eta/d\gamma_2\,|_{\eta=0} > 0$, and*

(c) *$g(\lambda, \gamma_2) = \beta G\gamma_2(1 + o(1))$ as $|\lambda| + \gamma_2 \to 0$, where G is defined in the table* 1.

Proof. The proof relies on finding the asymptotic formula for $\eta(\gamma, \lambda)$ as $|\lambda| + |\gamma| \to 0$. The function $s(\lambda)$ is chosen in such a way that the main terms in $\eta(\gamma, \lambda)$ are linear in γ_i and the coefficients standing before γ_i have the same singularity with respect to λ. This allows us to solve the equation $\eta = 0$.

We divide the trajectory (limit cycle) Γ into three parts as follows. Let C_1 and C_2 be certain constants not depending on λ and γ such that $x_4 < C_1\,|\lambda|$ and $y_4 < C_2\,|\lambda|$. Define

$$\begin{aligned} \Gamma^{(1)} &= \Gamma \cap \{x < C_1\,|\lambda|,\ y < C_2\,|\lambda|\}, \\ \Gamma^{(2)} &= (\Gamma \setminus \Gamma^{(1)}) \cap \{\dot{x} \geqslant 0\}, \\ \Gamma^{(3)} &= (\Gamma \setminus \Gamma^{(1)}) \cap \{\dot{x} < 0\} \end{aligned} \qquad (44)$$

(here $\dot{x} = x(\lambda - x + y)$). Denote by $\Gamma_0^{(i)}$, $i = 1, 2, 3$, the corresponding pieces of the curve $\Gamma_0 = H^{-1}(H(P_5))$. The curves $\Gamma^{(i)}$ $(\Gamma_0^{(i)})$, $i = 2, 3$, can be represented as a graphs of some functions

$$\{y = y^{(i)}(x)\} \qquad (\text{resp. } \{y = y_0^{(i)}(x)\}). \qquad (45)$$

The cases $-1<\alpha+\beta<0$ and $\alpha+\beta<-1$ need a separate analysis. We start with the first one.

D.1. $-1<\alpha+\beta<0$. (Here $\lambda<0$ and we replace λ with $-\lambda$, $\eta=-1$, $s(\lambda)=\lambda/\ln(1/\lambda)$, $\gamma_1<0$, $\gamma_2>0$.)

We shall estimate successively the increment of H along each $\Gamma^{(i)}$, $\Delta H\,|_{\Gamma^{(i)}}$, $i=1,2,3$. Let $i=1$. We need some information about the piece $\Gamma^{(1)}$ of the trajectory Γ.

LEMMA 5.5. *If $\lambda+|\gamma|$ is sufficiently small and $(x,y)\in\Gamma^{(1)}$ then*

$$\lambda x^\alpha y^\beta=(\beta+1)^{-1}+O\left(\frac{|x|}{\lambda}+\frac{|y|}{\lambda}+|\gamma|+\lambda\right).$$

Proof. See the Appendix.

By this lemma

$$\Delta H\,|_{\Gamma^{(1)}}=I_1^{(1)}+I_2^{(1)},$$

where

$$I_1^{(1)}=\frac{\gamma_1\lambda}{\beta\ln\lambda}\int_{\Gamma^{(1)}}x^{\alpha-1}y^\beta\,dx,$$

$$|I_2^{(1)}|=\left|\gamma_2\int_{\Gamma^{(1)}}x^{\alpha-1}y^\beta R(-x-\lambda,y)\,dx\right|$$

$$\leqslant\text{const.}\,\gamma_2\int_{\lambda^{-(\beta+1)/\alpha}}^{\lambda}x^{-1}\lambda^{-1}\lambda^2\,dx\leqslant\text{const.}\,\gamma_2\lambda\ln(1/\lambda)$$

and

$$\lim_{|\gamma|,\lambda\to0}\frac{\lambda}{\ln(1/\lambda)}\int_{\Gamma^{(1)}}x^{\alpha-1}y^\beta\,dx$$

$$=\lim_{\lambda\to0}\frac{\lambda}{\ln(1/\lambda)}\frac{1}{\beta+1}\int_{D\lambda^{-(\beta+1)/\alpha}}^{C_1\lambda}x^{-1}\lambda^{-1}\,dx$$

$$=-\frac{\alpha+\beta+1}{\alpha(\beta+1)}.$$

Here $(D\lambda^{-(\beta+1)/\alpha}, C_2\lambda)$ is the point of intersection of $\Gamma^{(1)}$ with the line $y=C_2\lambda$ and C_1 and C_2 are the constants defining $\Gamma^{(1)}$ (see (44)). Therefore

$$\Delta H\,|_{\Gamma^{(1)}}=\gamma_1\frac{\alpha+\beta+1}{\alpha\beta(\beta+1)}(1+o(1))+\gamma_2 o(1). \tag{46}$$

Now, let us consider the pieces $\Gamma^{(2)}$ and $\Gamma^{(3)}$. We estimate the corresponding functions $y^{(i)}$, $i=2, 3$ (Eq. (45)).

LEMMA 5.6. *There exists a constant K depending only on α and β such that*

(a) $K^{-1}x^{-\alpha/(\beta+1)} \leqslant y^{(2)}(x) \leqslant Kx^{-\alpha/(\beta+1)}$,

(b) $K^{-1}x^{-(\alpha+1)/\beta} \leqslant y^{(3)}(x) \leqslant Kx^{-(\alpha+1)/\beta}$,

(c) $|y^{(2)}-y_0^{(2)}|\,(x) \leqslant K(|\gamma_1|\,\lambda/\ln(1/\lambda)+\gamma_2 x^{-\alpha/(\beta+1)})$,

(d) $|x^{(3)}-y_0^{(3)}|\,(x) \leqslant K(|\gamma_1|\,x^{-1-(\alpha+1)/\beta}\ln^{-1}(1/\lambda)+\gamma_2 x^{-(\alpha+1)/\beta})$,

(e) Γ has the first order tangency with the line $\{x=1\}$ at the point P_5.

Proof. See the Appendix.

We estimate now the increment ΔH on $\Gamma^{(2)}$ and on $\Gamma^{(3)}$. We write

$$\Delta H\,|_{\Gamma^{(i)}} = \int_{\Gamma^{(i)}} \frac{dH}{dt}\,dt$$

$$= \int_{\Gamma_0^{(i)}} \frac{dH}{dt}\,dt + \int_{(\Gamma^{(i)}-\Gamma_0^{(i)})} \frac{dH}{dt}\,dt = I_1^{(i)} + I_2^{(i)}, \tag{47}$$

$i=2, 3$. We compute $I_1^{(i)}$, $i=2, 3$, using the points (a) and (b) of Lemma 5.6. Then

$$I_1^{(2)} = \frac{\gamma_1\lambda}{\ln(\lambda)}\, O\left(\int_{\lambda^{-(\beta+1)/\alpha}}^{1} x^{\alpha-1-(\alpha\beta/(\beta+1))}\,dx\right)$$

$$+\gamma_2 \int_{\Gamma^{(2)}_{0,\lambda=0}} x^{\alpha-1}y^{\beta}R(-x, y)\,dx\,(1+o(1))$$

and

$$I_1^{(3)} = \frac{\gamma_1\lambda}{\ln\lambda}\, O\left(\int_{\lambda}^{1} x^{\alpha-1-(\alpha+1)}\,dx\right)$$

$$+\gamma_2 \int_{\Gamma^{(3)}_{0,\lambda=0}} x^{\alpha-1}y^{\beta}R(-x, +y)\,dx\,(1+o(1)).$$

Thus

$$I_1^{(2)} + I_1^{(3)} = \gamma_2 \int_{H_0=\;-1/\beta(\beta+1)} x^{\alpha-1}y^{\beta}R(-x, y)\,dx + o(|\gamma|) \tag{48}$$

as $\lambda+|\gamma| \to 0$.

We use Lemma 5.6 to estimate $I_2^{(i)}$, $i=2, 3$. If $i=2$ then

$$|I_2^{(2)}| \leqslant \text{const.} \int_{\lambda^{-(\beta+1)/\alpha}}^{1} x^{\alpha-1} \left[\frac{\gamma_1 \lambda}{\ln \lambda} x^{-\alpha(\beta-1)/(\beta+1)} + \gamma_2 x^{-\alpha}\right]$$
$$\times \left[\frac{\gamma_1 \lambda}{\ln \lambda} + \gamma_2 x^{-\alpha/(\beta+1)}\right] dx \leqslant \text{const.} \left(\frac{\gamma_1}{\ln \lambda} + \gamma_2\right)^2. \tag{49}$$

If $i=3$ then

$$|I_2^{(3)}| \leqslant \text{const.} \int_{\lambda}^{1} x^{\alpha-1} \left[\frac{\gamma_1 \lambda}{\ln \lambda} x^{-(\alpha+1)(\beta-1)/\beta} + \gamma_2 x^{-(\alpha+1)((\beta-1)/\beta)+2}\right]$$
$$\times \left[\frac{\gamma_1 \lambda}{\ln \lambda} x^{-((\alpha+1)/\beta)-1} + \gamma_2 x^{-(\alpha+1)/\beta}\right] dx$$
$$= \text{const.} \int_{\lambda}^{1} x^{-2} \left[\frac{\gamma_1 \lambda}{\ln \lambda} + \gamma_2 x^2\right]\left[\frac{\gamma_1 \lambda}{\ln \lambda} x^{-1} + \gamma_2\right] dx$$
$$\leqslant \text{const.} \left(\frac{\gamma_1}{\ln \lambda} + \gamma_2\right)^2.$$

On the basis of (46), (48)–(50) we get

$$\eta(\gamma, \lambda) = F_1 \gamma_1 + F_2 \gamma_2 + o(|\gamma|) \qquad \text{as} \quad \lambda + |\gamma| \to 0,$$

where $F_i > 0$, $i=1, 2$. The equation $\eta=0$ has the solution $\gamma_1 = \beta G \gamma_2(1+o(1))$, where G is given in the Table I. From this and from (38) we obtain the formula (12) in Theorem 2.

It remains to compute the expression $d\eta/d\gamma_2$. Using (43) we easily find

$$\left.\frac{d\eta}{d\gamma_2}\right|_{\eta=0} = F_2 - o(1) > 0.$$

This completes the proof of Proposition 5.1 and Theorem 2c in our case.

D.2. $\alpha+\beta < -1$. (Here $\alpha<0$, $\beta>0$, $\lambda>0$, $\eta=-1$, $\gamma_1>0$, $s(\lambda) = \lambda^{-\alpha-\beta}$, or $\lambda^2 \ln(1/\lambda)$ or λ^2).

As in the previous case we start with the computation of the contribution arising from $\Gamma^{(1)}$ to ΔH. To do this we look at the behaviour of the curve $\Gamma^{(1)}$ near the singular point $P_3 = (\lambda, 0)$ (see Fig. 6).

As $\lambda \to 0$ the curve $\Gamma^{(1)}$ tends to the two separatrices of the saddle P_3. It is seen after extending the coordinates and time $x' = x/\lambda$, $y' = y/\lambda$, $t' = \lambda t$.

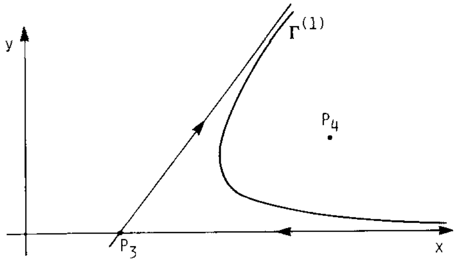

FIGURE 6

We obtain the system

$$\dot{x} = x(1 - x + y),$$
$$\dot{y} = y\left(-\frac{\alpha}{\beta} - \frac{\gamma_1 s(\lambda)}{\beta\lambda} + \frac{\alpha+1}{\beta}x - \frac{\alpha}{\beta+1}y + \gamma_2\lambda R(1-x, y)\right)$$

considered in the domain $\{0 < x < C_1,\ 0 < y < C_2\}$. This system converges to the conservative system with $H(x, y) = x^\alpha y^\beta(y/(\beta+1) + (1-x)/\beta)$ as a first integral as $\lambda \to 0$. The cycle Γ tends to the curve $H(x, y) = H(\lambda^{-1}, \lambda^{-1} - 1) = -\lambda^{-\alpha-\beta-1}(1 + o(1))/\beta(\beta+1)$. Because $\lambda^{-\alpha-\beta-1} \to 0$ as $\lambda \to 0$ the curve $\Gamma^{(1)}$ tends to the separatrices of the saddle P_3. If $\lambda + |\gamma| \to 0$ then the separatrices of the saddle P_3 tend to the lines $y = 0$ and $y = (\beta+1)(x-\lambda)/\beta$.

Therefore, in the integral formula for $\Delta H|_{\Gamma^{(1)}}$ (see (40)) we have $x^{\alpha-1} < \lambda^{\alpha-1}$ (since $x > \lambda$) and the other terms y^β and $y^\beta R$ are bounded and small. Due to this and due to the closeness of $\Gamma^{(1)}$ to the separatrices of P_3 we have

$$\begin{aligned}\Delta H|_{\Gamma^{(1)}} &= -\frac{\gamma_1 s(\lambda)}{\beta}\left(\frac{\beta+1}{\beta}\right)^\beta \int_\lambda^{C_1\lambda} x^{\alpha-1}(x-\lambda)^\beta\, dx\,(1 + o(1)) \\ &\quad + \gamma_2\left(\frac{\beta+1}{\beta}\right)^\beta \frac{1}{\beta^2(\beta+2)}\int_\lambda^{C_1\lambda} x^{\alpha-1}(x-\lambda)^{\beta+2}\, dx\,(1+o(1)) \\ &= I_1^{(1)} + I_2^{(1)}, \end{aligned} \tag{51}$$

where

$$I_2^{(1)} = \begin{cases} \gamma_2 O(\lambda^{\alpha+\beta+2}) & \text{if } -2 < \alpha+\beta < -1 \\ \gamma_2\left(\frac{\beta+1}{\beta}\right)^\beta \frac{1+o(1)}{\beta^2(\beta+2)}\int_\lambda^{C_1\lambda} x^{\alpha-1}(x-\lambda)^{\beta+2}\, dx & \text{if } \alpha+\beta \leqslant -2 \end{cases} \tag{52}$$

and C_1 is the constant defining $\Gamma^{(1)}$ (Eq. (44)).

The next step is the computation of $\Delta H|_{\Gamma^{(i)}}$, $i = 2, 3$. Before doing it we formulate the lemma analogous with Lemma 5.6.

LEMMA 5.7. *There exists a constant K depending only on α and β such that*

(a) $K^{-1}x \leqslant y^{(2)}(x) \leqslant Kx$,

(b) $K^{-1}x^{-(\alpha+1)/\beta} \leqslant y^{(3)}(x) \leqslant Kx^{-(\alpha+1)/\beta}$,

(c) $|y^{(2)} - y_0^{(2)}|\,(x) \leqslant K[\gamma_1 s(\lambda) + \gamma_2 s(x)]$,

(d) $|y^{(3)} - y_0^{(3)}|\,(x) \leqslant K[\gamma_1 s(\lambda)\, x^{-1-(\alpha+1)/\beta} + \gamma_2 x^{-(\alpha+1)/\beta}]$,

(e) *Γ has the first order tangency with the line $\{x=1\}$.*

Proof. See the Appendix.

Now, we are ready to complete the proof of Proposition 5.1 in the case $\alpha+\beta < -1$. We compute $\Delta H\,|_{\Gamma^{(i)}}$, $i=2, 3$, using the splitting (47). We have

$$I_1^{(2)} = \frac{-\gamma_1 s(\lambda)}{\beta}\left(\frac{\beta+1}{\beta}\right)^{\beta} \int_{C_1\lambda}^{1} x^{\alpha-1}(x-\lambda)^{\beta}\,dx\,(1+o(1))$$

$$+\gamma_2 \int_{\Gamma_0^{(2)}} x^{\alpha-1} y^{\beta} R(\lambda - x, y)\,dx\,(1+o(1)),$$

where C_1 is the constant defining $\Gamma^{(i)}$ (Eq. (44)). If $i=3$ then

$$I_1^{(3)} = \gamma_1 s(\lambda)\, O\left(\int_{\lambda}^{1} x^{\alpha-1-\alpha-1}\,dx\right)$$

$$+\gamma_2 \int_{\Gamma_0^{(3)}} x^{\alpha-1} y^{\beta} R(\lambda - x, y)\,dx$$

$$= \gamma_1 s(\lambda)\, O(\lambda^{-1}) + \gamma_2 \int_{\Gamma_{0,\lambda=0}^{(3)}} x^{\alpha-1} y^{\beta} R(-x, y)\,dx\,(1+o(1)).$$

By (51), (52) and above we obtain

$$I_1^{(2)} + I_1^{(3)} + \Delta H\,|_{\Gamma^{(1)}}$$

$$= -\frac{\gamma_1 s(\lambda)}{\beta}\,\lambda^{\alpha+\beta}\left(\frac{\beta+1}{\beta}\right)^{\beta}\int_1^{\infty} x^{\alpha-1}(x-1)^{\beta}\,dx\,(1+o(1))$$

$$+\gamma_2(1+o(1))\begin{cases} \displaystyle\int_{H_0=H_0(P_5)} x^{\alpha-1}y^{\beta}R(-x,y)\,dx & (\alpha+\beta > -2), \\[2ex] \displaystyle\frac{\ln(1/\lambda)}{\beta^2(\beta+2)}\left(\frac{\beta+1}{\beta}\right)^{\beta} & (\alpha+\beta = -2), \\[2ex] \displaystyle\frac{\lambda^{\alpha+\beta+2}}{\beta^2(\beta+2)}\left(\frac{\beta+1}{\beta}\right)^{\beta}\int_1^{\infty} x^{\alpha-1}(x-1)^{\beta+2}\,dx & (\alpha+\beta < -2), \end{cases} \tag{53}$$

where

$$\int_1^{\infty} x^{\alpha-1}(x-1)^{\beta}\,dx=\int_0^1 x^{-(\alpha+\beta+1)}(1-x)^{\beta}\,dx$$
$$=B(-\alpha-\beta,\beta+1)=\frac{\Gamma(-\alpha-\beta)\,\Gamma(\beta+1)}{\Gamma(1-\alpha)}$$

(see [5]). The remainder of $\Delta H=\eta(\gamma,\lambda)$ is estimated as follows:

$$|I_2^{(2)}|\leqslant \text{const.}\int_{\lambda}^{1} x^{\alpha-1}[\gamma_1 s(\lambda)\,x^{\beta-1}+\gamma_2 x^{\beta+1}]$$
$$\times[\gamma_1 s(\lambda)+\gamma_2 s(x)]\,dx\leqslant \text{const.}\,|\gamma|^2\,s(\lambda)\,\lambda^{\alpha+\beta} \tag{54}$$

and

$$|I_2^{(3)}|\leqslant \text{const.}\int_{\lambda}^{1} x^{-2}[\gamma_1 s(\lambda)+\gamma_2 x^2]$$
$$\times[\gamma_1 s(\lambda)\,x^{-1}+\gamma_2]\,dx\leqslant \text{const.}(\gamma_1 s(\lambda)/\lambda+\gamma_2)^2$$
$$\leqslant \text{const.}\,|\gamma|^2\,s(\lambda)\,\lambda^{\alpha+\beta} \tag{55}$$

(the latter estimate is the same as the estimate (50)). Now from (47), (53)–(55) we have

$$\eta(\gamma,\lambda)=s(\lambda)\,\lambda^{\alpha+\beta}[-\gamma_1 F_1+\gamma_2 F_2+o(|\gamma|)] \tag{56}$$

for $\lambda+|\gamma|\to 0$, where $F_i>0$, $i=1,2$. We see that the equation $\eta=0$ has the solution $\gamma_1=\beta G\gamma_2(1+o(1))$, where G is defined in Table I. From this and from (38) we obtain the formula (12) in Theorem 2.

To answer the question as to whether Γ is stable or not, we calculate

$$\left.\frac{d\eta}{d\gamma_2}\right|_{\eta=0}=\begin{array}{ll} F_2(\alpha+\beta+2)+o(1) & (\alpha+\beta>-2),\\ F_2+o(1) & (\alpha+\beta=-2),\\ o(1) & (\alpha+\beta<-2).\end{array}$$

Thus we need only consider the case $\alpha+\beta<-2$. We use the formula (41). Hence we must fix the sign of the following integral

$$J=\oint_{\Gamma}\frac{-\gamma_1\lambda^2+\gamma_2 z^2}{xz}\,dx,$$

where $z=\lambda-x+y$. The calculations repeat the proof of the formula (56).

Finally we get

$$J = -\gamma_1 O(\lambda) + \gamma_2 \oint_{\Gamma_0} \frac{z\,dx}{x}\,(1 + o(1)) > 0.$$

This completes the proof of Proposition 5.1 for $\alpha + \beta < -1$.

E. *Γ is not Large and Far from Singular Points*

Here we work in the system of coordinates defined by (34) with $\gamma_2 = |\mu_1|$. We have $\lambda = \operatorname{sign} \mu_1$ and $s(|\lambda|) = 1$.

Since Γ is far from the singular points of $V_{\lambda,\gamma}$ according to continuous dependence on parameters and Lemma 5.3 we have

$$\eta(h) = \iint_{H \leqslant h} x^{\alpha-1} y^{\beta-1}(-\gamma_1 + \gamma_2 z^2)\,dx\,dy + O(|\gamma|^2) = I(h, \gamma) + O(|\gamma|^2) \tag{57}$$

and

$$\eta'(h) = \oint_{H=h} \frac{-\gamma_1 + \gamma_2 z^2}{xz}\,dx + O(|\gamma|^2) = \frac{\partial}{\partial h} I(h, \gamma) + O(|\gamma|^2), \tag{58}$$

where $z = \lambda + \eta x + y$, $\lambda = \operatorname{sign} \mu_1$, $\eta = \operatorname{sign}(\alpha + \beta)$. The last identity in (58) follows from (57) and from the identity $dx\,dy = x^{-\alpha} y^{1-\beta} z^{-1}\,dx\,dH$.

Define the function

$$g(h) = \frac{I_1(h)}{I_2(h)} = \iint_{H \leqslant h} x^{\alpha-1} y^{\beta-1} z^2\,dx\,dy \Big/ \iint_{H \leqslant h} x^{\alpha-1} y^{\beta-1}\,dx\,dy. \tag{59}$$

The equation $\eta = 0$ has the solution $\gamma_1 = g(h)\gamma_2 + O(|\gamma|^2)$. Moreover, by (58)

$$\eta'\,|_{\eta=0} = \gamma_2 I_2(h)\, g'(h) + O(|\gamma|^2). \tag{60}$$

The point (c) of Theorem 2 in our case follows from (60) and from the following result proof of which we leave to the last section.

PROPOSITION 5.2. *$g'(h) > 0$ for $h > h_{\min}$.*

At this moment we can assure the proofs of Theorems 1 and 2 as complete. What remains is the construction of the conjugacy homeomorphisms (7) between the systems (8) and (28) (or between (8) and (8) with different (α, β)). Such constructions have been done by Bogdanov in [8] for another family, and it is not difficult to repeat his proof.

6. Proof of Monotonicity of g

This section is crucial for the whole paper. We investigate the integrals along the curves which depend on parameters irrationally. At first we reduce the problem to the case $\beta \geqslant 1$.

Lemma 6.1. *If Proposition 5.2 holds for $\beta \geqslant 1$ then in the other cases it is also true.*

Proof. The idea relies on the extension of the action of the group $S(3)$ described in Section 3B onto all vector fields $V_{\lambda,\gamma}$ of the form (35) (with $\lambda = \pm 1$, $s(|\lambda|) = 1$ and $V_3 \equiv 0$).

The action of σ_1: $(x, y) \to (y, x)$ remains unchanged (see (24)). From the proof of Proposition 3.2 it follows that the vector field transformed by σ_1 is orbitally equivalent to a vector field of the form (35). This defines the transformation $\bar{\sigma}_1$ on the space of vector fields of the form (35).

Before describing the action of σ_2 we look at the consequences of the action of $\bar{\sigma}_2$ on the (α, β)-plane (27). The mapping $\bar{\sigma}_2$: $(\alpha, \beta) \to (\alpha/\beta, 1/\beta)$ transforms the domain $\{\alpha, \beta > 0\}$ into itself and if $0 < \alpha \leqslant \beta < 1$ then $\beta' = 1/\beta > 1$ and $\alpha' = \alpha/\beta \leqslant 1 < \beta'$. The domain VII (Fig. 1) is transformed into itself and (as above) if $0 < \beta < 1$ then $\beta' > 1$. The domains $\{\alpha, \beta < 0, \alpha + \beta > -1\}$ and $\{\alpha > 0, \alpha + \beta < -1\}$ are transformed one into other. Thus the composition $\bar{\sigma}_1 \circ \bar{\sigma}_2$ transforms the domain VIII $\{\alpha \leqslant \beta < 0, \alpha + \beta > -1\}$ onto $\{\beta \geqslant 1, \alpha + \beta < -1\}$. Hence the second order terms can be pushed to the case $\beta \geqslant 1$.

We define the action of σ_2 applied to the vector fields (35). We put σ_2 of the form

$$\sigma_2 = \sigma_3 \circ \sigma_4 \circ \sigma_5,$$

where σ_3, $\sigma_5 \in \mathscr{G}_0$ (see (17)), $\sigma_4(x, y) = (x, y + k\hat{x} + P_\lambda(\hat{x}))$, $\hat{x} = x + \lambda$, $k = (\beta + 1)/\beta$, $P_\lambda(\hat{x}) = P = a\hat{x}^2 + b\hat{x}\lambda + c\lambda^2$.

First, we define σ_4 applied to the vector field of the form

$$W_\lambda: \begin{cases} \dot{x} = x(\hat{x} + y + \varphi_1(\hat{x}, y)) \\ \dot{y} = y\left(-\dfrac{\alpha}{\beta}\lambda - \dfrac{\alpha + 1}{\beta}x - \dfrac{\alpha}{\beta + 1}y + \rho\lambda\hat{x} + \varphi_2(\hat{x}, y)\right), \end{cases}$$

where $\varphi_1 = q_1\hat{x}^2 + q_2\hat{x}y + q_3 y^2$, $\varphi_2 = q_4\hat{x}^2 + q_5\hat{x}y + q_6 y^2$, λ is small and $\rho = O(1)$. (Every 3-jet of the vector field V_μ has such a representation). We assume that x, $\hat{x}$ and y are of order λ. We define P to satisfy the condition: 3-jet of $\sigma_{3*} W_\lambda \circ \sigma_3^{-1}$ is tangent to the axes. The perturbation $W_\lambda - V_{\lambda,0}$ of the vector field $V_{\lambda,0}$ (see (36)) is of order λ^3. So we compute the homogenous (with respect to the homogenous filtration in the variables $\hat{x}$

and λ) part of $\sigma_{3*} W_\lambda \circ \sigma_3^{-1}$ of order 3 which is not tangent to the axes. (The part of lower order is tangent.)

The nontangent part is $Q(\hat{x}, \lambda)\, \partial_y$, where

$$\begin{aligned} Q = & -(k\hat{x} + P)\left[(1-k)(\hat{x}-\lambda) + \frac{\alpha}{\beta+1} P + \rho\lambda\hat{x} + \varphi_2(1,-k)\,\hat{x}^2\right] \\ & + (k(\hat{x}-\lambda) + P'(\hat{x}-\lambda))[(1-k)\,\hat{x} - P + \varphi_1(1,-k)\,\hat{x}^2] \\ = & \frac{-1}{\beta} P'(\hat{x}-\lambda)\,\hat{x} + P\left(\lambda - \frac{\alpha+\beta}{\beta}\hat{x}\right) - k\rho\lambda\hat{x}^2 \\ & + k\varphi_1(1,-k)\,\hat{x}^2(\hat{x}-\lambda) - k\varphi_2(1,-k)\,\hat{x}^3 \end{aligned}$$

(mod $|(\hat{x}, \lambda)|^4$). The condition $Q = 0$ implies $c = 0$, $b = 0$, and

$$\frac{\beta+2}{\beta} a = k\rho + k\varphi_1(1,-k),$$

$$\frac{\alpha+\beta+2}{\beta} a = -k\varphi_1(1,-k) + k\varphi_2(1,-k).$$

Therefore we have the following necessary condition,

$$Q_1 = (\alpha+\beta+2)\,\rho + (\alpha+2\beta+4)\,\varphi_1(1,-k) - (\beta+2)\,\varphi_2(1,-k) = 0$$

to solve the equation $Q = 0$. This condition is compatible with the condition $I_1 \neq 0$, where

$$\begin{aligned} I_1 = & \frac{\alpha(\alpha+\beta+1)}{\beta(\beta+1)} \\ & \times\left[\frac{\alpha+2}{\beta+1} q_1 - \frac{\alpha+1}{\beta+1} q_2 + \frac{\alpha+1}{\beta} q_3 + \frac{\beta+1}{\beta} q_4 - q_5 + \frac{(\alpha+1)(\beta+2)}{\alpha\beta} q_6\right] \end{aligned}$$

(see (20)). Indeed, the noncompatibility of the conditions $Q_1 = 0$ and $I_1 \neq 0$ means that the hypersurfaces $Q_1 = 0$ and $I_1 = 0$ are coincident. Obviously it is not possible for $(\alpha+\beta+2)\,\rho \neq 0$. In the opposite case the comparision of the coefficients standing before q_i's gives the same assertion.

We define now the transformation $\sigma_5 \in \mathscr{G}_0$. First, we use the dilations of variables to put the vector field $V_{\lambda,\gamma}$ (35) to the same with λ small, $\eta = 1$ and $\gamma_2 = 1$ and represent it in the form W_λ with $\varphi_1 = 0$ and $\varphi_2 = R$. Next, we use the transformation $g_{\rho,\lambda} \in \mathscr{G}_0$ of the form given in (17) to satisfy the condition $Q_1 = 0$. This transformation depends on ρ as well as on λ although the dependence on λ is very weak. From the proof of Proposition 3.2 follows that such a transformation exists. (The compatibility of the conditions $Q_1 = 0$ and $I_1 \neq 0$ is here essential). The composition of the above two transformations defines σ_5.

The considerations presented above imply that $V = (\sigma_4 \circ \sigma_5)(V_{\lambda,\gamma})$ is the vector field with first order ($\leqslant 3$) terms tangent to the axes. We choose this part and define the transformation σ_3 to obtain the vector field (35) by applying Proposition 3.2 and the transformation (34). This completes the proof of Lemma 6.1.

In the sequel we assume that $\beta \geqslant 1$. By (58) and (59) we need to show the inequality

$$(I_1' - gI_2')(h) = \oint_{H=h} \left(z - \frac{g}{z}\right) \frac{dx}{x} = \int_{x_1}^{x_2} \left[z_2 - z_3 - g\left(\frac{1}{z^2} - \frac{1}{z_3}\right)\right] \frac{dx}{x} > 0, \quad (62)$$

where $z_i = z_i(x) = y_i(x) + \eta x + \lambda$ and $\{y = y_i(x)\}$, $i = 2, 3$, are branches of the curve $\{H = h\}$ lying above and below the line $z = 0$, respectively, and $x_1 < x_2$ are x-components of the intersection of the line $z = 0$ with the curve $H = h$ (see Fig. 7). The idea of the proof of Inequality (62) is following. We estimate the functions $|z_i(x)|$ from below as in Fig. 7. The estimating functions are chosen explicitly integrable. This and some crude estimate of $g(h)$ permit us to succeed.

Let $x_1 < x_0 < x_2$ be such a point that the function $(z_2 z_3)(x) < 0$ takes its minimal value at x_0. We formulate below five lemmas from which Proposition 5.3 follows.

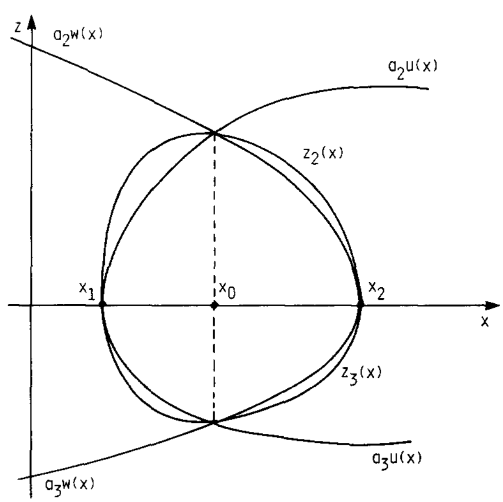

FIGURE 7

LEMMA 6.2. *There exist $\gamma > 0$ and $x_5 > x_0$ such that*

$$z_2(x) \geqslant a_2[(x_5^\gamma - x_1^\gamma)^2 - (x_5^\gamma - x^\gamma)^2]^{1/2} = a_2 u(x), \qquad z_3(x) \leqslant a_3 u(x), \tag{63}$$

for $x_1 \leqslant x \leqslant x_0$, where $a_i = z_i(x_0)/u(x_0)$, $i = 2, 3$.

LEMMA 6.3. *There exist $\delta > 0$ and $x_6 > x_2$ such that*

$$z_2(x) \geqslant \tilde{a}_2[(x_6^\delta - x^\delta)^2 - (x_6^\delta - x_2^\delta)^2]^{1/2} = \tilde{a}_2 w(x), \qquad z_3(x) \leqslant \tilde{a}_3 w(x) \tag{64}$$

for $x_0 < x < x_2$, where $\tilde{a}_i = z_i(x_0)/w(x_0)$, $i = 2, 3$.

LEMMA 6.4. *We have the inequality*

$$0 < -g(h)/(z_2 z_3)(x_0) \leqslant \tfrac{1}{3}$$

for $h > h_{\min}$ and $\beta \geqslant 1$.

LEMMA 6.5. *We have the inequality*

$$\int_{x_1}^{x_0} \left[u(x) - \frac{u^2(x_0)}{3u(x)} \right] \frac{dx}{x} > 0. \tag{65}$$

LEMMA 6.6. *We have the inequality*

$$\int_{x_0}^{x_2} \left[w(x) - \frac{w^2(x_0)}{3w(x)} \right] \frac{dx}{x} > 0. \tag{66}$$

From these lemmas the inequality (62) follows. Indeed by Lemma 6.2, 6.4, and 6.5 we have

$$\begin{aligned}
&\int_{x_1}^{x_0} \left[z_2 - z_3 - g(h) \left(\frac{1}{z_2} - \frac{1}{z_3} \right) \right] \frac{dx}{x} \\
&\quad \geqslant \int_{x_1}^{x_0} \left[(a_2 - a_3)\, u(x) - g(h) \left(\frac{1}{a_2} - \frac{1}{a_3} \right) \frac{1}{u(x)} \right] \frac{dx}{x} \\
&\quad = (a_2 - a_3) \int_{x_1}^{x_0} \left[u(x) + \frac{g(h)\, u^2(x_0)}{(z_2 z_3)(x_0)\, u(x)} \right] \frac{dx}{x} \\
&\quad \geqslant (a_2 - a_3) \int_{x_1}^{x_0} \left[u(x) - \frac{u^2(x_0)}{3u(x)} \right] \frac{dx}{x} > 0.
\end{aligned}$$

Analogously we verify that $\int_{x_0}^{x_2} [z_2 - z_3 - g(h)((1/z_2) - (1/z_3))](dx/x) > 0$. This ends the proof of Proposition 5.2.

Proof of Lemma 6.2. The cases $\alpha > 0$ and $\alpha < 0$ we consider separately. Let $\alpha > 0$. We define the functions

$$f_i(x, z) = z^2 - z_i^2(x_0)\, u^2(x)/u^2(x_0), \qquad i = 2, 3, \tag{67}$$

where $u^2(x) = (x_5^\gamma - x_1^\gamma)^2 - (x_5^\gamma - x^\gamma)^2$.

The idea is the following: We have two parameters x_5 and γ to control the functions f_2 and f_3. If $\gamma = 1$ then the curve $f_2 = 0$ forms an ellipse about which we do not know whether it is placed as Lemma 6.2 asserts. It is because of its convexity. If one increases the parameter γ then the curve $f_2 = 0$ becomes nonconvex and the possibility to satisfy (63) rises. There is no direct method to prove the inequalities (63) because we have no explicit formulas expressing the functions $z_2(x)$ and $z_3(x)$. We omit this difficulty by investigating the directions of the vector field $V_{\lambda,0}$ along the curves $f_i = 0$.

It is enough to show that for certain x_5 and γ the following assertion holds (see Fig. 8).

The functions

$$g_i = \frac{d}{dt} f_i(x, z), \; i = 2, 3,$$

restricted to the curves $\{f_i = 0, \; (-1)^i z \geqslant 0\}$ parametrized by $x \in [x_1, x_0]$ have the following behaviour:

> $g_i(x_1) = 0$, $(-1)^i g_i(x_0) < 0$, and g_i have only one change of sign in the interval (x_1, x_0).

(The derivative d/dt is understood to be the derivative along the vector field $V_{\lambda,0}$.)

We have

$$\frac{d}{dt} f_i(x, z) = 2z[\dot{z} - \gamma z_i^2(x_0)\, x^\gamma (x_5^\gamma - x^\gamma)/u^2(x_0)] = 2z\xi_i(x, z).$$

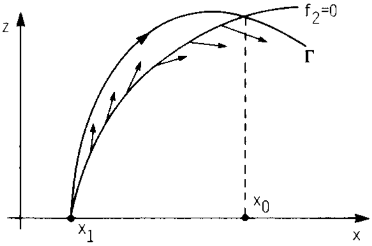

FIGURE 8

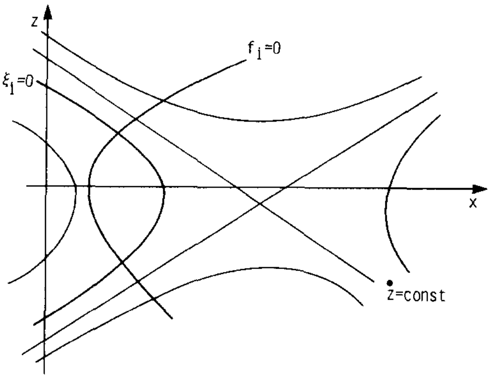

FIGURE 9

The curves $\{\dot{z}=\text{const.}\}$ are hyperbolas given by the equations $x=\varphi(z)$, where φ has only one maximum at a point $\bar{z}\leqslant 0$, $\varphi'(0)\leqslant 0$ (see Fig. 9). Now, assume that $x_5^\gamma > 2x_0^\gamma$. Then the function $x^\gamma(x_5^\gamma - x^\gamma)$ is increasing for $0<x<x_0$ from what follows, that the curve $\xi_i=0$ is given by the equation $x=\psi(z)$, where $\psi(-1)=\psi(1/\beta)=0$ and ψ has only one maximum at a point $\tilde{z}<0$ (Fig. 9).

Because the function $u(x)$ is increasing for $x\in(x_1, x_0)$ it suffices to prove that

$$\xi_i(x_1, 0)>0 \qquad \text{and} \qquad \xi_i(x_0, z_i(x_0))<0, \quad i=2, 3.$$

We shall tend with γ to $+\infty$ and shall choose x_5 such that $(x_0/x_5)^\gamma \to 0$. We have

$$\begin{aligned}\xi_i(x, z) &= \dot{z} - \gamma z_i^2(x_0)\,\frac{x^\gamma}{x_0^\gamma - x_1^\gamma}\,\frac{x_5^\gamma - x^\gamma}{2x_5^\gamma - x_1^\gamma - x_0^\gamma}\\ &= \dot{z} - \frac{1}{2}\,\gamma z_i^2(x_0)\left(\frac{x}{x_0}\right)^\gamma (1+o(1)).\end{aligned} \tag{68}$$

Therefore

$$\xi_i(x_1, 0) = \dot{z} - \frac{1}{2}\gamma\left(\frac{x_1}{x_0}\right)^\gamma O(1) \to \dot{z}(x_1, 0)>0 \qquad \text{as} \quad \gamma\to\infty.$$

and

$$\xi_i(x_0, z_i(x_0)) = \dot{z} - \frac{1}{2}\gamma z_i^2(x_0)(1+o(1)) \to -\infty \qquad \text{as} \quad \gamma\to\infty.$$

Let $\alpha<0$. In this case we introduce the functions f_i (67) and we strive to

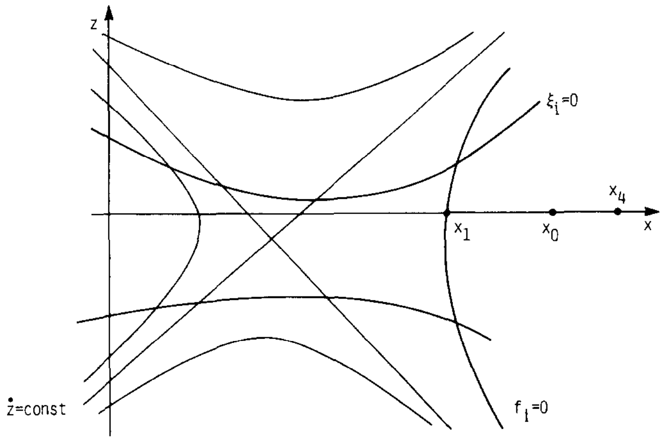

FIGURE 10

prove the same assertion as for $\alpha>0$. But the situation now is more complicated. The curves $\dot{z}=$ const. are distributed as in Fig. 10. Thus the curve $\xi_i=0$ takes one of two forms (both consisting of two components): either the components are vertical and resemble the curves $\dot{z}=0$ (they are given by the equations $x=\varphi_1(z)$ and $x=\varphi_2(z)$, where φ_1 (φ_2) has unique maximum (minimum)) or they are horizontal curves (given by the equations $z=\psi_1(x)$ and $z=\psi_2(x)$, where ψ_i are of the form as φ_i) (see Fig. 10). In the first case we analyze only the right component $x=\varphi_2(z)$.

We shall show the following assertion.

The curves $\xi_i=0$ and $f_i=0$ have exactly one point of intersection in the domain $\{x_1<x<x_0, (-1)^i z>0\}$ and

$$\xi_i(x_1, 0)>0 \qquad \text{and} \qquad \xi_i(x_0, z_i(x_0))<0, \quad i=2, 3. \tag{69}$$

We verify the last property analogously for $\alpha>0$. To prove other assertions we write

$$\{f_i=0\}=\Big\{z=z_i(x_0)((x^\gamma-x_1^\gamma)/(x_0^\gamma-x_1^\gamma))^{1/2}(1+o(1))$$
$$=z_i(x_0)\left(\frac{x}{x_0}\right)^{\gamma/2}(1+o(1))\Big\}$$

and

$$\{\xi_i=0\}=\left\{\dot{z}(x, z)=\frac{1}{2}\gamma z_i^2(x_0)\left(\frac{x}{x_0}\right)^\gamma(1+o(1))\right\},$$

where

$$\dot{z}(x, z) = \frac{\alpha+\beta+1}{\beta(\beta+1)}(x-1)(x-x_4) - \frac{(\alpha+\beta+1)(\beta-1)}{\beta(\beta+1)}(x-x_4)\,z - \frac{\alpha}{\beta+1}\,z^2.$$

From this it is seen that we must solve the equation

$$-a(x-1)(x-x_4) = b\gamma(x/x_0)^{\gamma}. \tag{70}$$

asymptotically as $\gamma \to \infty$. Here $a > 0$, $b > 0$, $1 < x_1 < x < x_0$, $x_4 = \alpha/(\alpha+\beta+1)$. There are two possibilities either (i) $x_4 \leqslant x_0$ or (ii) $x_4 > x_0$. Equation (70) has a unique solution $x = x_4 + o(1)$ in the interval (x_1, x_0) in the case (i) and $x = x_0(1 - O(\gamma^{-1}\ln\gamma))$ in the case (ii).

This completes the proof of Lemma 6.2.

Remark 6.1. In [45] another family of vector fields with a limit cycle has been considered. In the analysis the function of the type (59) has appeared and the estimate

$$z_i^2 \geqslant \text{const.}(x_1^{\gamma} - x^{\gamma}), \qquad x \in (x_1, x_0).$$

has been used. Probably this estimate holds in our case too (at least for $\alpha > 0$) but the author does not know the proof. (In [45] it has been proven that the integral in (65) with $u(x)$ replaced by $x_1^{\gamma} - x^{\gamma}$ is positive.)

Proof of Lemma 6.3. We prove the lemma for $\delta = 1$. Analogously to the proof of Lemma 6.2 we introduce the functions

$$f_i(x, z) = z^2 - \tilde{a}_i^2(x_6 - x)^2 + \tilde{a}_i^2(x_6 - x_2)^2, \qquad i = 2, 3$$

and

$$\left(\frac{d}{dt} f_i\right)(x, z) = 2z(z + \tilde{a}_i x(x_6 - x)) = 2z\xi_i(x, z),$$

where ξ_i are the second order polynomials. As in the previous proof it suffices to show the following assertion.

In the domain $D_i = \{H < 0,\ (-1)^i z > 0\}$ the quadratic curve $\xi_i = 0$ and the hyperbola $f_i = 0$ have only one point of intersection provided $f_i(\{\xi_i = z = 0\}) < 0$.

First, note that the hyperbola $f_i = 0$ is symmetric with respect to the x axis, the (quadratic) curve $\xi_i = 0$ contains the points P_1 and P_2 on the boundaries of D_i (see Fig. 5) and that $f_i(\{\xi_i = z = 0\}) < 0$. These facts allow us to state that the number of the required intersections is at most 2. Next, at the point $\bar{P}$ of the intersection of $f_i = 0$ with the boundary of D_i we have $(-1)^i\, df_i/dt(\bar{P}) > 0$ (the curve $f_i = 0$ intersects the trajectories of the vector field $V_{\lambda,0}$ transversally). Hence $\xi_i(\bar{P}) > 0$ and our assertion holds.

Now, we choose x_6 to fulfill the condition $f_i(\{\xi_i = z = 0\}) < 0$ which is equivalent to the following one:

If the curve $H = h$ is given by the equality $z^2 = d(x_2 - x)(1 + o(1))$ in a neighbourhood of the point $(x_2, 0)$ then

$$\frac{\partial f_i}{\partial x}(x_2, 0) = 2z_i^2(x_0)\,\frac{x_2}{x_2 - x_0}\,\frac{x_6 - x_2}{2x_6 - x_2 - x_0} < d, \qquad i = 2, 3.$$

This condition is satisfied if x_6 is sufficiently close to x_2.

Lemma 6.3 is complete.

Proof of Lemma 6.4. We need to prove the inequality

$$\iint_{H \leqslant h} x^{\alpha - 1} y^{\beta - 1}\left(z^2 + \tfrac{1}{3}(z_2 z_3)(x_0)\right) dx\, dy$$
$$= \int_{x_1}^{x_2} x^{\alpha - 1}\left[\int_{y_1(x)}^{y_2(x)} y^{\beta - 1}\left(z^2 + \tfrac{1}{3}(z_2 z_3)(x_0)\right) dy\right] dx \leqslant 0.$$

We show that

$$f(x, h) = \int_{y_3(x)}^{y_2(x)} y^{\beta - 1}\left(z^2 + \tfrac{1}{3}(z_2 z_3)(x)\right) dy \leqslant 0 \tag{71}$$

from which the assertion of Lemma 6.4 follows (because $(z_2 z_3)(x_0) \leqslant (z_2 z_3)(x)$).

We simplify the problem by changing $y_i(x)/(-\lambda - \eta x)$ and $z_i(x)/(-\lambda - \eta x)$ with y_i and z_i, respectively, $i = 2, 3$. Then $f(x, h)$ changes to

$$\begin{aligned} f(h) &= \int_{y_3}^{y_2} y^{\beta - 1}\left((y - 1)^2 + \tfrac{1}{3} z_2 z_3\right) dy \\ &= y_2^{\beta}\left(\frac{y_2^2}{\beta + 2} - \frac{2y_2}{\beta + 1} + \frac{1}{\beta} + \frac{z_2 z_3}{3\beta}\right) - y_3^{\beta}\left(\frac{y_3^2}{\beta + 2} - \frac{2y_3}{\beta + 1} + \frac{1}{\beta} + \frac{z_2 z_3}{3\beta}\right), \end{aligned} \tag{72}$$

where $z_i = y_i - 1$, $i = 2, 3$, and y_2 and y_3 satisfy the equation

$$y_2^{\beta}\left(\frac{y_2}{\beta + 1} - \frac{1}{\beta}\right) = y_3^{\beta}\left(\frac{y_3}{\beta + 1} - \frac{1}{\beta}\right), \qquad y_3 < y_2. \tag{73}$$

If $\beta = 1$ then $z_2 = -z_3$ and $f(h) = 0$. We assume further that $\beta > 1$.

If $h=0$ then $y_2=(\beta+1)/\beta$, $y_3=0$, $z_2=1/\beta$, $z_3=-1$, and

$$f(0)=\left(\frac{\beta+1}{\beta}\right)^{\beta}\left[\frac{(\beta+1)^2}{\beta^2(\beta+2)}-\frac{2}{\beta}+\frac{1}{\beta}-\frac{1}{3\beta^2}\right]$$
$$=\left(\frac{\beta+1}{\beta}\right)^{\beta}\frac{1}{\beta^2}\left[\frac{1}{\beta+2}-\frac{1}{3}\right]<0 \tag{74}$$

for $\beta>1$.

The function $f(h)$ can be treated as a function of the variable z_2, $f(h)=\tilde{f}(z_2)$. Obviously $\tilde{f}(0)=0$.

LEMMA 6.7. *We have*

$$\tilde{f}(z_2)=O(z_2^5) \qquad \text{as} \quad z_2\to 0$$

and

$$\frac{d^5\tilde{f}}{dz_2^5}(0)=\frac{-4}{135}(\beta-1)(2\beta+1)<0$$

for $\beta>1$.

Proof. The property $\tilde{f}(z_2)=O(z_2^4)$ follows from (72) because $z_2/z_3\to -1$ as $z_2\to 0$ and then

$$\tilde{f}(z_2)=\int_{-z_2}^{z_2}\left(z^2-\frac{1}{3}z_2^2\right)dz+O(z_2^4)=O(z_2^4).$$

Next from the expansion

$$y^{\beta}\left(\frac{y}{\beta+1}-\frac{1}{\beta}\right)=\frac{-1}{\beta(\beta+1)}+\frac{1}{2}z^2+\frac{1}{3}(\beta-1)\,z^3+\frac{1}{8}(\beta-1)(\beta-2)\,z^4+\cdots,$$

one can solve Eq. (73) in the form

$$z_3=-z_2-\tfrac{2}{3}(\beta-1)\,z_2^2+O(z_2^3).$$

Hence we have

$$z_2+z_3=\tfrac{2}{3}(\beta-1)\,z_3z_2(1+O(z_2)) \qquad \text{as} \quad z_2\to 0.$$

We substitute this expression into the formula (76) for $\tilde{f}'(z_2)$. Standard calculations show that

$$\tilde{f}'(z_2)=-\frac{4}{27}(\beta-1)(2\beta+1)\,z_2^4(1+O(z_2)) \qquad \text{as} \quad z_2\to 0$$

which completes the proof of Lemma 6.7.

To prove the inequality $\tilde{J}(z_2)<0$ for $0<z_2<1/\beta$ we show the following property which gives the inequality (71) together with (74) and Lemma 6.7.

LEMMA 6.8. *Let $\beta>1$. If $\tilde{J}(z_2)=0$ for $z_2>0$ then $\tilde{J}'(z_2)<0$.*

Proof. If $\tilde{J}(z_2)=0$ then by (72) and (73)

$$\begin{aligned}
&\left(\frac{y_3}{\beta+1}-\frac{1}{\beta}\right)\left(\frac{y_2^2}{\beta+2}-\frac{2y_2}{\beta+1}+\frac{1}{\beta}+\frac{z_2z_3}{3\beta}\right)\\
&\quad-\left(\frac{y_2}{\beta+1}-\frac{1}{\beta}\right)\left(\frac{y_3^2}{\beta+2}-\frac{2y_3}{\beta+1}+\frac{1}{\beta}+\frac{z_2z_3}{3\beta}\right)\\
&\quad=\frac{z_2-z_3}{3\beta(\beta+1)(\beta+2)}\left[2(\beta-1)\,z_2z_3-3(z_2+z_3)\right]=0
\end{aligned}$$

or simply

$$z_2+z_3=\tfrac{2}{3}(\beta-1)\,z_2z_3. \tag{75}$$

From (73) we have $dz_3/dz_2=(y_2/y_3)^{\beta-1}\,z_2/z_3$. We differentiate (72) using the latter and (73),

$$\begin{aligned}
\tilde{J}'(z_2)&=y_2^{\beta-1}(z_2^2+\tfrac{1}{3}z_2z_2)+\frac{z_3}{3\beta}(y_2^\beta-y_3^\beta)\\
&\quad-\left(\frac{y_2}{y_3}\right)^{\beta-1}\frac{z_2}{z_3}\left[y_3^{\beta-1}\left(z_3^2+\frac{1}{3}z_2z_3\right)-\frac{z_2}{3\beta}(y_2^\beta-y_3^\beta)\right]\\
&=y_2^{\beta-1}\left[\frac{2}{3}z_2(z_2-z_3)+\frac{z_3y_2}{3\beta}\left(1-\left(\frac{y_3}{y_2}\right)^\beta\right)+\frac{z_2^2y_3}{3\beta z_3}\left(\left(\frac{y_2}{y_3}\right)^\beta-1\right)\right]\\
&=\frac{y_2^{\beta-1}(z_2-z_3)}{z_3\left(\frac{y_2}{\beta+1}-\frac{1}{\beta}\right)\left(\frac{y_3}{\beta+1}-\frac{1}{\beta}\right)}\\
&\quad\times\left[\frac{2}{3}z_2z_3\left(\frac{y_2}{\beta+1}-\frac{1}{\beta}\right)\left(\frac{y_3}{\beta+1}-\frac{1}{\beta}\right)-\frac{z_3^2y^2}{3\beta(\beta+1)}\left(\frac{y_2}{\beta+1}-\frac{1}{\beta}\right)\right.\\
&\quad\left.-\frac{z_2^2y_3}{3\beta(\beta+1)}\left(\frac{y_3}{\beta+1}-\frac{1}{\beta}\right)\right]\\
&=\frac{y_2^{\beta-1}(z_2-z_3)}{3z_2(z_2-1/\beta)(z_2-1/\beta)}\\
&\quad\times\{2(\beta-1)\,\beta^{-1}(z_2z_3)^2-(3\beta-1)\,\beta^{-2}(z_2z_3)(z_2+z_3)\\
&\quad+\beta^{-2}(z_2+z_3)^2\}.
\end{aligned}\tag{76}$$

Substituting (75) into the braces in the last expression we get

$$\bar{f}'(z_2)=\frac{y_2^{\beta-1}(z_2-z_3)(\beta-1)}{3\beta^2 z_3(z_2-1/\beta)(z_3-1/\beta)}\,\frac{2}{9}\,(2\beta+1)(z_2z_3)^2.$$

This expression is nonpositive because $z_3<0$, $z_3<z_2<1/\beta$, and $\beta>1$. Lemma 6.4 is complete.

Proof of Lemma 6.5. Let us note at the beginning that the sign of the integral (61) is invariant under the changes $\gamma\to\gamma_1>0$, $x, x_i\to Cx$, Cx_i, $C>0$. Therefore we can assume that $\gamma=1$, $x_1=1$, $x_5=2$. We use the substitution $\cos\varphi=2-x$, $\cos\varphi_0=2-x_0\in(0,1)$. We obtain the following integral instead of the one in (65),

$$\begin{aligned} f(\varphi_0)&=\int_0^{\varphi_0}\frac{\sin^2\varphi-(\sin^2\varphi_0)/3}{2-\cos\varphi}\,d\varphi\\ &=\sin\varphi_0+2\varphi_0-\left(3+\frac{1}{3}\sin^2\varphi_0\right)\int_0^{\varphi_0}\frac{d\varphi}{2-\cos\varphi}.\end{aligned}$$

Denote by f_1 the following function (of the same sign as f),

$$f_1(\varphi)=\frac{\sin\varphi+2\varphi}{3+(\sin^2\varphi)/3}-\int_0^{\varphi}\frac{d\psi}{2-\cos\psi}.$$

Obviously $f_1(0)=0$. We compute the derivative of f_1,

$$f_1'(\varphi)=f_2(\varphi)/3(2-\cos\varphi)(3+(\sin^2\varphi)/3)^2,$$

where

$$\begin{aligned} f_2(\varphi)&=3(2-\cos\varphi)(2+\cos\varphi)(3+\tfrac{1}{3}\sin^2\varphi)\\ &\qquad-2(\sin\varphi+2\varphi)\sin\varphi\cos\varphi(2-\cos\varphi)-3(3+\tfrac{1}{3}\sin^2\varphi)^2\\ &=4\sin\varphi\{[\sin\varphi-\varphi\cos\varphi-\tfrac{1}{3}\sin^3\varphi]\\ &\qquad+[\varphi(1-\cos\varphi)+\sin\varphi(1-\cos\varphi)-\varphi\sin^2\varphi]\}\\ &=4\sin\varphi(f_3(\varphi)+f_4(\varphi)).\end{aligned}$$

Let us consider $f_3(\varphi)$. Obviously $f_3(0)=0$ and

$$f_3'(\varphi)=\sin\varphi(\varphi-\tfrac{1}{2}\sin(2\varphi))\geqslant 0.$$

Hence $f_3(\varphi)>0$.

Consider f_4. We have

$$f_4(\varphi)=(1-\cos\varphi)(\sin\varphi-\varphi\cos\varphi)=(1-\cos\varphi)f_5(\varphi),$$

where $f_5(0)=0$ and $f'_5(\varphi)=\varphi \sin \varphi \geqslant 0$. Thus $f_5>0$ and then $f>0$ which finishes the proof of Lemma 6.5.

Proof of Lemma 6.6. Analogously to the beginning of the proof of Lemma 6.5 we can assume that $\delta=1$, $x_2=1$, and $x_6=2$. We use the substitution $\cosh \varphi=2-x$, $\cosh \varphi_0=2-x_0 \in (1,2)$. Then instead of the integral in (66) we get

$$\begin{aligned} f(\varphi_0) &= \int_0^{\varphi_0} \frac{\sinh^2 \varphi-(\sinh^2 \varphi_0)/3}{2-\cosh \varphi}\, d\varphi \\ &= -\sinh \varphi_0 - 2\varphi_0 + \left(3-\tfrac{1}{3}\sinh^2 \varphi_0\right) \int_0^{\varphi_0} \frac{d\varphi}{2-\cosh \varphi}. \end{aligned}$$

We introduce the function (of the same sign as f)

$$f_1(\varphi)=\int_0^{\varphi} \frac{d\psi}{2-\cosh \psi}-\frac{\sinh \varphi+2\varphi}{3-(\sinh^2 \varphi)/3}.$$

Obviously $f_1(0)=0$. Differentiating f_1 we obtain

$$f'_1(\varphi)=f_2(\varphi)/3(2-\cosh \varphi)(3-(\sinh^2 \varphi)/3)^2,$$

where

$$\begin{aligned} f_2(\varphi) = {} & 3(3-\tfrac{1}{3}\sinh^2 \varphi)^2-3(3-\sinh^2 \varphi)(3-\tfrac{1}{3}\sinh^2 \varphi) \\ & -2(\sinh \varphi+2\varphi)\sinh \varphi \cosh \varphi(2-\cosh \varphi)=4\sinh \varphi \\ & \times \{[\varphi(1-\cosh \varphi)+\tfrac{1}{2}\varphi \sinh^2 \varphi] \\ & +[\sinh \varphi(1-\cosh \varphi)+\tfrac{1}{2}\varphi \sinh^2 \varphi] \\ & +[\sinh \varphi-\varphi \cosh \varphi+\tfrac{1}{3}\sinh^3 \varphi]\} \\ = {} & 4\sinh \varphi\{f_3(\varphi)+f_4(\varphi)+f_5(\varphi)\}. \end{aligned}$$

We have

$$f_3(\varphi)=\tfrac{1}{2}\varphi(\cosh \varphi-1)^2 \geqslant 0.$$

Next

$$f_4(\varphi)=\tfrac{1}{2}(\cosh \varphi-1)(\varphi+\varphi \cosh \varphi-2\sinh \varphi)=\tfrac{1}{2}(\cosh \varphi-1)\, f_6(\varphi),$$

where

$$f'_6(\varphi)=1-\cosh \varphi+\varphi \sinh \varphi, \qquad f''_6(\varphi)=\varphi \cosh \varphi \geqslant 0.$$

Because $f_6(0)=f'_6(0)=0$ we have $f_4(\varphi)>0$ for $\varphi>0$.

Finally $f_5(0)=0$ and

$$f'_5(\varphi)=\sinh\varphi(\sinh\varphi\cosh\varphi-\varphi)=\sinh\varphi(\tfrac{1}{2}\sinh(2\varphi)-\varphi)\geqslant 0$$

and hence $f_5>0$. Lemma 6.6. is proved.

Appendix

Proof of Lemma 5.5. If $\gamma=0$ then we have the following equation for $\Gamma_0^{(1)}$,

$$x^\alpha y^\beta\left(\frac{\lambda+x}{-\beta}+\frac{y}{\beta+1}\right)=H(P_5)=\frac{-1+O(\lambda)}{\beta(\beta+1)}, \tag{77}$$

where $-1<\beta<0$, $x<C_1\lambda$, and $y<C_2\lambda$ (see (44)). Therefore for $\gamma=0$ Lemma 5.5 is proved.

If $\gamma\neq 0$ then we consider the derivative of the function H along the vector field $V_{\lambda,\gamma}$,

$$\left|\frac{dH}{dt}\right|<C\lambda(\gamma_1\ln^{-1}\lambda+\gamma_2\lambda)\,H, \tag{78}$$

where C does not depend on x, y, λ (here $\gamma_1<0$ and $H>0$).

Now, we look at the position of the upper endpoints S_1 and S_1^0 of the curves $\Gamma^{(1)}$ and $\Gamma_0^{(1)}$, respectively. We are interesting in the estimation of the distance between the endpoints. If we know that the distance is small we shall show the required estimate of $\Delta H\,|_{\Gamma^{(1)}}$ and of the distance between the lower endpoints S_2 and S_2^0 of $\Gamma^{(1)}$ and $\Gamma_0^{(1)}$, respectively.

The estimate of the distance between S_1 and S_1^0 can be derived from points (a) and (c) of Lemma 5.6. Points (a) and (c) of Lemma 5.6 will be proved independently from Lemma 5.5. Therefore one has

$$S_1=(D\lambda^{-(\beta+1)/\alpha},C_2\lambda),\qquad S_1^0=(D^0\lambda^{-(\beta+1)/\alpha},C_2\lambda), \tag{79}$$

where $D=D^0(1+O(|\gamma|))$ and $|H(S_1)-H(S_1^0)|<\text{const.}\,|\gamma|$.

Because $\dot{x}=x(-\lambda-x+y)$ the time at which the trajectory $\Gamma^{(1)}$ starting from the point S_1 meets the point S_2 is bounded as follows:

$$t<\text{const.}\,\lambda^{-1}\ln(1/\lambda).$$

Therefore solving Inequality (78) we obtain

$$H(t)=H(S_1)\exp[O(|\gamma_1|+\gamma_2\lambda\ln(1/\lambda))]=H(S_1)(1+O(|\gamma|)).$$

From this and from (77), Lemma 5.5 follows.

Proof of Lemma 5.6. If $\gamma = 0$ then from (77) we have

$$H(P_5) + \frac{x^\alpha y^{\beta+1}}{\beta} < \frac{x^\alpha y^{\beta+1}}{\beta+1} = H(P_5) + \frac{x^\alpha (x+\lambda)\, y^\beta}{\beta}$$
$$< H(P_5), \qquad (x, y) \in \Gamma_0^{(2)},$$

$P_5 = (1, 1+\lambda)$ (because $x + \lambda < y$ and $-1 < \beta < 0$) and

$$H(P_5) > H(P_5) - \frac{x^\alpha y^{\beta+1}}{\beta+1} = \frac{x^\alpha (x+\lambda)\, y^\beta}{-\beta}$$
$$> C x^\alpha y^\beta \left(\frac{y}{\beta+1} - \frac{x+\lambda}{\beta} \right) = CH(P_5),$$

$(x, y) \in \Gamma_0^{(3)}$ for certain $C > 0$ (because $x > C_1 \lambda$ and $y < x + \lambda$). From this inequalities (a) and (b) of Lemma 5.6 for $\gamma = 0$ follow.

If γ is not 0 then to estimate the curve $\Gamma^{(2)}$ from above we consider the curve U: $y = \varphi(x) = Kx^{-\alpha/(\beta+1)} - Lx$. We look at the direction of the vector field $V_{\lambda,\gamma}$ along U. We have

$$\frac{d}{dt}(y - \varphi(x)) \bigg|_{y=\varphi(x)}$$
$$= \frac{1}{\beta+1} \left[\frac{\alpha}{\beta} \lambda y + \frac{\alpha+\beta+1}{\beta} xy + (\alpha+\beta+1)\, Lx(-\lambda - x + y) \right]$$
$$+ \frac{|\gamma_1|\, \lambda}{\beta \ln \lambda}\, y + \gamma_2\, yR(-\lambda - x, y) = \psi_\gamma(x, y).$$

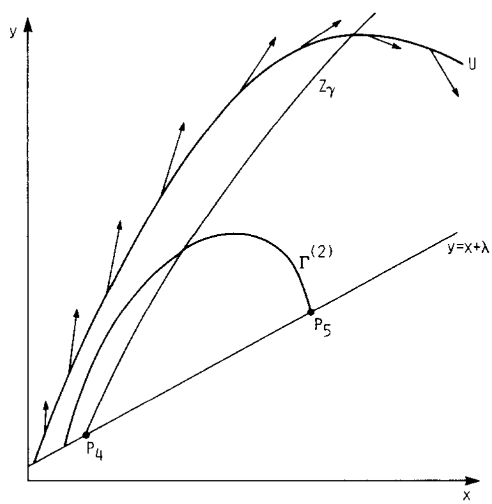

FIGURE 11

Denote by Z_γ the curve $\psi_\gamma(x, y) = 0$. At the point of intersection of U with Z_γ the vector field $V_{\lambda,\gamma}$ is tangent to U. Z_0 is the hyperbola and Z_γ is the small perturbation of Z_0 (see Fig. 11). Therefore the unique point Q of intersection of U with Z_γ satisfy the estimate $|Q| > \text{const.} > 0$, where const. depends only on K and L. The curve $\Gamma^{(2)}$ differs little from the curve $\Gamma_0^{(2)}$ in the domain $|(x, y)| > \text{const.}$ From the above and from Fig. 11 it is seen that one can choose K and L in such a way that $y^{(2)}(x) < \varphi(x) < Kx^{-\alpha/(\beta+1)}$.

To estimate the function $y^{(2)}$ from below we consider the function

$$\varphi(x) = K(x - L\lambda^{-(\beta+1)/\alpha})^{-\alpha/(\beta+1)}, \qquad K < 1$$

and analyze the expression

$$\begin{aligned}\frac{d}{dt}(y - \varphi(x))\bigg|_{y=\varphi(x)} &= y\left[\left(\frac{\alpha}{\beta(\beta+1)} + \frac{\gamma_1}{\beta \ln \lambda}\right)\lambda + \frac{\alpha+\beta+1}{\beta(\beta+1)}\,x\right. \\ &\quad \left. + \frac{\alpha}{\beta+1}\,\frac{L\lambda^{-(\beta+1)/\alpha}}{x - L\lambda^{-(\beta+1)/\alpha}}\,(y - x - \lambda) + \gamma_2 R(-x-\lambda, y)\right] = y\psi(x, y).\end{aligned}$$

It is not difficult to show that $\psi(x, y) < 0$ along the curve $y = \varphi(x)$ (see Fig. 12) in the domain $y > C_2\lambda$ for suitably choosen L. From this the estimate $y^{(2)}(x) > K^{-1}x^{-\alpha/(\beta+1)}$ can be easily derived. Point (a) is complete.

To estimate $y^{(3)}$ we consider the curve $U\colon x = \varphi(y)$, $\varphi(y) = Ky^{-\beta/(\alpha+1)} - Ly$ and investigate the expression

$$\frac{d}{dt}(x - \varphi(y))\bigg|_{x=\varphi(y)} = \psi_\gamma(x, y),$$

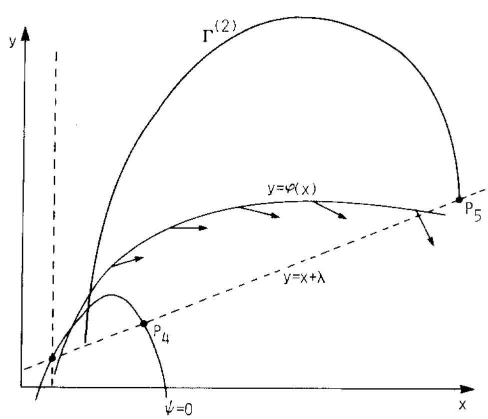

FIGURE 12

where

$$\psi_0(x, y) = x\left(\frac{(\alpha+\beta+1)\,y}{(\alpha+1)(\beta+1)} - \frac{\lambda}{\alpha+1}\right) + \frac{\alpha+\beta+1}{\alpha+1}\, Ly\left(\frac{\alpha+1}{\beta}\, x - \frac{\alpha y}{\beta+1} + \frac{\alpha}{\beta}\,\lambda\right)$$

is a quadratic function and ψ_γ is a small perturbation of ψ_0. Further proof of the inequality $y^{(3)}(x) > Kx^{-(\alpha+1)/\beta}$ proceeds in the same way as the proof of the upper estimate of the function $y^{(2)}$.

Using the above function $\varphi(y)$ and Lemma 5.5, one can show the estimate of $y^{(3)}$ from above. Observe that from estimate (79) and from formula (46) one has the following assertion about the lower endpoint S_2 of the curve $\Gamma^{(1)}$ (and $\Gamma^{(3)}$, too),

$$S_2 = (C_1\lambda, E\lambda^{-(\alpha+1)/\beta}),$$

where $E = (\beta+1)^{-1/\beta} + o(1)$ as $\lambda + |\gamma| \to 0$. Then by this and by the above argument (see Fig. 13),

$$y^{(3)}(x) < \max[2(\beta+1)^{-1/\beta}, 2]\; x^{-(\alpha+1)/\beta}$$

which ends the proof of point (b) of Lemma 5.6.

To show estimates (c) and (d) we find differential equations which are satisfied by the functions $\tilde{y}^{(i)} = y^{(i)} - y_0^{(i)}$, $i = 2, 3$.

Let $i = 2$. Using inequality (a) and the fact that $x^{-\alpha/(\beta+1)} \gg x$ for small x from (35)–(37) we have

$$\begin{aligned}\frac{d\tilde{y}^{(2)}}{dx} &= \frac{\dot{y}}{\dot{x}}(x, y^{(2)}) - \frac{\dot{y}}{\dot{x}}(x, y_0^{(2)})\\ &= -\frac{\alpha}{\beta+1}\frac{\tilde{y}^{(2)}}{x}(1 + k(x)) + \frac{\gamma_1\lambda}{\ln\lambda}O(x^{-1}) + \gamma_2 O(x^{-2\alpha/(\beta+1)-1}), \quad (80)\end{aligned}$$

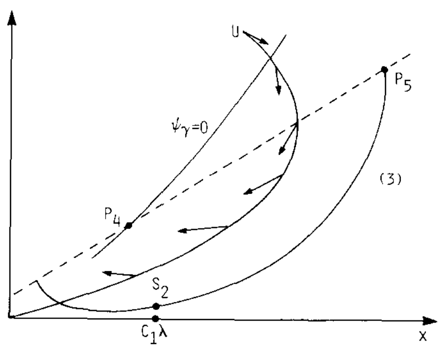

FIGURE 13

where $k(x)=o(1)$ as $x \to 0$. We treat (80) as a linear inhomogeneous equation of $\tilde{y}^{(2)}$ with initial value $\tilde{y}^{(2)}(1)=0$. (We substitute in the needed places of (80) the actual functions $y^{(2)}(x)$ or $y_0^{(2)}(x)$.)

The general solution of the homogenous equation is

$$\tilde{y}=Cx^{-\alpha/(\beta+1)}(1+o(1)).$$

Therefore the solution of the equation (80) is following

$$\begin{aligned}\tilde{y}^{(2)}(x)&=x^{-\alpha/(\beta+1)}\int_1^x x^{\alpha/(\beta+1)}\\&\quad\times\left[\frac{\gamma_1\lambda}{\ln\lambda}\,O(x^{-1})+\gamma_2 O(x^{-2\alpha/(\beta+1)-1})\right]dx\\&=\frac{\gamma_1\lambda}{\ln\lambda}\,O(1)+\gamma_2 O(x^{-\alpha/(\beta+1)})\end{aligned}$$

(because $\alpha/(\beta+1)<0$).

Let $i=3$. Here we use inequality (b) of Lemma 5.6 and the fact that $x^{-(\alpha+1)/\beta} \ll x$ for small x. We have the following differential equation for $\tilde{y}^{(3)}=y^{(3)}-y_0^{(3)}$,

$$\begin{aligned}\frac{d\tilde{y}^{(3)}}{dx}&=\frac{\alpha+1}{-\beta}\,\frac{\tilde{y}^{(3)}}{x}\,\frac{x+\alpha\lambda/(\alpha+1)}{x+\lambda}\,(1+o(1))\\&\quad+\gamma_1 s(\lambda)\,O(x^{-((\alpha+1)/\beta)-2})+\gamma_2 O(x^{-((\alpha+1)/\beta)}).\end{aligned}\tag{81}$$

We recall that $x>C_1\lambda$ in our domain (see (44)). We treat Eq. (81) as an inhomogenous linear equation with the initial value $\tilde{y}^{(3)}(1)=0$. The general solution of the homogenous equation is $\tilde{y}=Cx^{-\alpha/\beta}(x+\lambda)^{-1/\beta}\,(1+o(1))=Cx^{-(\alpha+1)/\beta}\,O(1)$ (since $x>C_1\lambda$). From this, estimate (d) of Lemma 5.6 can be derived by integration (as for $i=2$).

Point (e) holds because at $P_5=\Gamma\cap\{x=1\}$ $\dot{x}(P_5)=0$, $d\dot{x}(P_5)\neq 0$ and $\dot{y}(P_5)\neq 0$. Lemma 5.6 is complete.

Proof of Lemma 5.7. The proof of points (b), (d), and (e) is the same as the proof of points (b), (d), and (e) of Lemma 5.6 and we do not repeat it.

To prove (a) let us observe that the curve $\Gamma^{(2)}$ lies below the separatrix of the saddle $P_3=(\lambda,0)$ (see Figs. 5 and 6). For $\gamma=0$ this separatrix is given by the equation $y=(\beta+1)(x-\lambda)/\beta$ and it is rather clear that $y^{(2)}(x)<[2(\beta+1)/\beta]\,x$ for arbitrary and small γ. The lower bound follows from the inequality $y^{(2)}(x)>x-\lambda>\text{const.}\,x$ which holds in the apprioriate domain containing $\Gamma^{(2)}$.

Point (c) we prove using the differential equation for $\tilde{y}^{(2)} = y^{(2)} - y_0^{(2)}$. At the beginning we find the equation for $\bar{y}^{(2)} = y^{(2)} - (\beta+1)(x-\lambda)/\beta$,

$$\frac{d\bar{y}^{(2)}}{dx} = \frac{\dot{y} - (\beta+1)\,\dot{x}/\beta}{\dot{x}}$$

$$= \frac{\bar{y}^{(2)}(-x - \alpha(\beta+1)^{-1}\,y) - \gamma_1 s(\lambda)\,\beta^{-1}y + \gamma_2 R(\lambda - x, y)}{x(\lambda - x + y)}.$$

Because $y \sim (\beta+1)(x-\lambda)/\beta$ in $\Gamma^{(2)}$ (for small x) $(-x - \alpha y/(\beta+1))/(\lambda - x + y) = -\alpha - \beta + \beta\lambda/(x-\lambda) + o(1)$ for $x > C_1\lambda$. Thus we have the following linear equation for $\tilde{y}^{(2)}$,

$$\frac{d\tilde{y}^{(2)}}{dx} = \left(-\frac{\alpha+\beta}{x} + \frac{\beta\lambda}{x(x-\lambda)}\right)\tilde{y}^{(2)}(1 + o(1)) + \gamma_1 s(\lambda)\ O(x^{-1}) + \gamma_2 O(x).$$

The general solution of the homgenous equation is

$$\tilde{y} = Cx^{-\alpha-\beta}\left(\frac{x-\lambda}{x}\right)^{\beta}(1 + o(1)) = Cx^{-\alpha-\beta}O(1) \qquad \text{for const. } \lambda < x \leqslant 1.$$

Therefore

$$|\tilde{y}^{(2)}| \leqslant \text{const. } x^{-\alpha-\beta}\int_x^1 (\gamma_1 s(\lambda)\,x^{\alpha+\beta-1} + \gamma_2 x^{\alpha+\beta+1})\,dx$$

$$\leqslant \text{const.}[\gamma_1 s(\lambda) + \gamma_2 s(x)].$$

This completes the proof of Lemma 5.7.

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